Homework 4 Solution

4.1 Inverse horseshoe.
Sketch the inverse map of the following horseshoe map in Fig. H4.1.

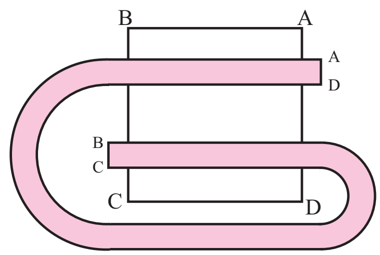

Figure H4.1: Horseshow preserving the 'vertical order'

Soln.
Pull the pink band straight, recording the portions of the band overlapping the white square. The square must be deformed continuously, so we can guess how the inverse map deforms the white square as illustrated in the following figure.

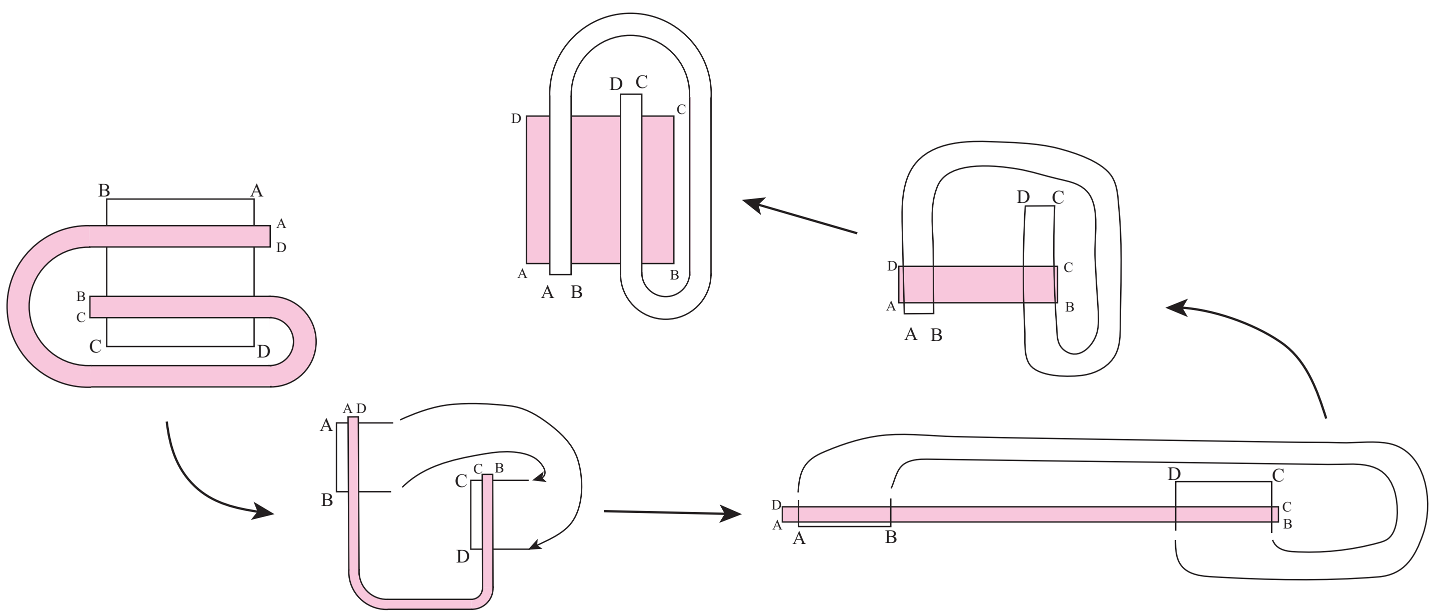

Figure H4.2: Horseshoe preserving the 'vertical order'

Thus, we see the inverse map is also an 'order-preserving' horseshoe.

4.2 Skewed tent maps and their entropy.

Consider the following generally skewed tent map (Fig. H4.3) ($\alpha \in (0,1)$) :

$$x \to f_\alpha(x) = \begin{cases} x/\alpha & \text{for } x \in [0,\alpha] \\ (1-x)/(1-\alpha) & \text{for } x \in [\alpha,1] \end{cases} . \tag{H4.1}$$

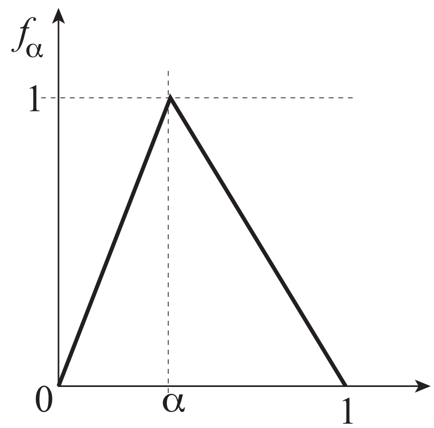

Figure H4.3: Skewed tent map

This interval dynamical system ($f_\alpha, [0,1]$) can have many distinct invariant measures, but only one of them is observable (i.e., reachable from a positive measure set of initial conditions). It is the flat measure $\mu = 1$ on $[0,1]$ for all α.

(1) Show that indeed this is an invariant measure.

Soln.

The easiest way is to check the Perron-Frobenius equation (31.9):

$$\frac{g(x_1)}{|F'(x_1)|} + \frac{g(x_2)}{|F'(x_2)|} = g(x). \tag{H4.2}$$

In our case, we have only to check that $g = 1$ satisfies this: $\alpha + 1 - \alpha = 1!$. The end.

(2) For this invariant measure compute the Kolmogorov-Sinai entropy h.

Soln.

The easiest way is to use Rokhlin's formula (31.1). In our case

$$h = \int_0^1 dx\,\log|F'(x)| = \int_0^\alpha dx\,\log(1/\alpha) + \int_\alpha^1 dx\,\log(1/(1-\alpha)) = -\alpha\log\alpha - (1-\alpha)\log(1-\alpha). \tag{H4.3}$$

In an alternative[1] approach we can use the standard definition of the Kolmogorov-Sinai entropy in terms of partitions (or generators). As you can easily show $\mathcal{A} = \{[0,\alpha],(\alpha,1]\}$ is a generator. This immediately gives the above formula. Besides, this approach tells you that the topological entropy of these maps is always $\log 2$, because it is given by counting

[1]Not in the Conway sense.

the number of pieces (= elements) in the partition $\mathcal{A} \vee f^{-1}\mathcal{A} \vee \cdots \vee f^{-(n-1)}\mathcal{A}$.

(3) (2) implies that if $\alpha \neq \beta$, then $(f_\alpha, 1, [0,1])$ and $(f_\beta, 1, [0,1])$ are not isomorphic ($\alpha, \beta \in (0,1)$). However, there is always a homeomorphism $\phi : [0,1] \to [0,1]$ that makes f_α and f_β conjugate for any $\alpha, \beta \in (0,1)$ pair. In particular, we can always find (α-dependent) ϕ such that ($f_{1/2}$ is the standard tent map)

$$\phi \circ f_\alpha = f_{1/2} \circ \phi. \tag{H4.4}$$

However, ϕ is, as you expect, a horrible continuous function like the devil's staircase.

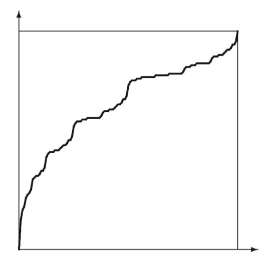

Figure H4.4: ϕ for $\alpha = 3/4$. [Fig. 2 of M Plakhotnyk, Topological conjugation of one dimensional maps, arXiv:1603.0690v1 (Mar, 2016)]

This means we can make an isomorphism ϕ (or its inverse) to map $(f_{1/2}, 1, [0,1])$ to $(f_\alpha, \mu_\alpha, [0,1])$ with $h_{\mu_\alpha}(f_\alpha) = \log 2$. Or we can find $(f_{1/2}, \nu, [0,1])$ with the same KS entropy as $(f_\alpha, 1, [0,1])$ as computed in (2): $h_\nu(f_{1/2}) = h_1(f_\alpha)$. Although a trivial question, explain that this means for a given (topological) dynamical system $(f_\alpha, [0,1])$ to have uncountably many distinct measure-theoretical dynamical systems.

Soln.
We know if the KS entropies are distinct, the dynamical systems cannot be isomorphic. Therefore, even if the dynamics (i.e., the map f) is the same, if two of its invariant measures μ and μ' have distinct KS entropies, then the measure-theoretical dynamical systems (f, μ, I) and (f, μ', I) are not isomorphic. We have just realized that for any f_α there is an invariant measure that gives the KS entropy that is equal to any number in $(0, \log 2]$. Thus, there must be uncountably many distinct invariant measures.

As is noted clearly in Story Line, which invariant measure we encounter depends on the initial condition. Everybody knows that the choice of the initial condition has nothing to do with the system dynamics (if inside the domain). You might say that we can observe only attracting periodic orbits or an absolutely continuous measures, so actually the dynamics enforces what we can encounter. This sounds a correct answer, but it totally ignores a deeper question why we do not observe Lebesgue measure zero sets. Certainly, this question is beyond dynamics/mechanics and is usually ignored by physicists (certainly, totally ignored by statistical physicists traditionally).

(4) For any $\alpha \in (0, 1)$ the topological entropy of f_α is $\log 2$ (again trivial but explain this).

Soln.

One answer was already given above. But sup $h = \log 2$ may be an easier answer.

4.3 Baker's map-symbolic dynamics coerrespondence

We have introduce a 01 coding of the point in the unit square $Q = [0, 1] \times [0, 1]$ for baker's map (see Fig. 27.2). However, distinct code sequences may correspond to the same point in Q. To characterize the points having this nonuniqueness, let us make an explicit coding formula for $(x, y) \in Q$.

Looking at Fig. 27.2, we see $(x, y) \in M_{i_0} \cap T^{-1}M_{i_1} \cdots T^{-n}M_{i_n}$ indicates that x-coordinate can be coded by this 'vertical striping'. If we code $a_{-k} = 1$ (or 0) according to $x \in T^{-k}M_1$ (or $T^{-k}M_0$), then the coding is $x \to a_0a_{-1} \cdots a_{-n} \cdots$. Analogously, for y we can use the 'horizontal layers': If we code $a_k = 1$ (or 0) according to $y \in T^kM_1$ (or $T^{-k}M_0$), then the coding is $y \to a_1a_2 \cdots a_n \cdots$.

(1) Compute x and y in terms of their (one-sided) sequences.

Soln.

The following formulas should be obvious:

$$x = \sum_{k=0}^{\infty} 2^{-k-1} a_{-k}, \tag{H4.5}$$

and

$$y = \sum_{k=1}^{\infty} 2^{-k} a_k. \tag{H4.6}$$

(2) Show that there are countably many points that correspond to more than one 01 sequences.

Soln.

This is just the question of identifying $0.01111111111 \cdots$ with $0.1000000000 \cdots$. The expressions for x and y surely give the same numbers. THus, needless to say, there are infintely may such points.