Homework 3 Solution

In the following I use 'ergodicity' in an intuitive sense: all the phase states (position + moving directions are evenly covered by a typical trajectory). We will seriously discuss it when we go to the second half of the course (a more serious part).

1. Consider a 2D billiard sketched in the figure Fig. H3.1

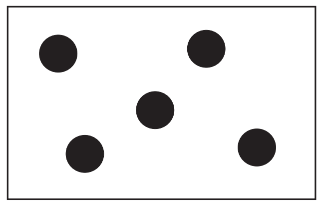

Figure H3.1: Rössler attractor

The rectangular table is $5D \times 8D$ and has 5 circular scatters of diameter D. At each collision roughly $-2\log_2 R$ bits of information about the phase point is lost. What is the information loss rate h (= the Kolmogorov-Sinai entropy = information loss/time) for this billiard table as a continuous time Hamiltonian system? You may assume[1] that the billiard is 'ergodic'.

Soln.
We can use the Abramov formula:

$$h = -2\log_2 R/\langle\tau\rangle, \tag{H3.1}$$

where $\langle\tau\rangle$ is the mean free time. The mean free time may be computed as explained in **17.16**,[2]

$$\langle\tau\rangle = 2\pi|Q|/2|\delta Q| = 2\pi(40 - (5\pi/4))D^2/10\pi D = (8 - \pi/4)D. \tag{H3.2}$$

Therefore, $(R = D/2)$

$$h = -2\log_2 R/(8 - \pi/4)D. \tag{H3.3}$$

2. In the famous 'Cloud' address[3] on the fundamental difficulty of classical physics, Lord Kelvin reconsidered the equipartition of energy to understand the specific heat ratio anomaly (that is, $\gamma = C_P/C_V > 1$ does not converge to 1 for large multiatomic gasses). Maxwell and Boltzmann assumed ergodicity (that is, all the phase points are evenly experienced by

[1]Provable.

[2]Those who are critical may demand the proof for more than 1 obstacle cases. In such cases, decompose the table into domains which contain one obstacle each and divide the integrals into the ones over individual domains. The numerator becomes the sum of the conditional averages of $\tau(r,\varphi)$ (in the individual domains) times the individual domain volume. If this is divided by the total table volume, it is the true average of τ, because generally $E(X) = \sum_B P(B)E(X|B)$.

[3]"Nineteenth-Century Clouds over the Dynamical Theory of Heat and Light" April 21, 1900.

molecules) to prove the equipartition of energy, so Kelvin attacked this hypothesis as the main cause of the trouble. He even reported 'numerical simulation results' (by his assistant Mr Anderson with a ruler and sheets of paper), claiming ergodicity is violated. He discussed, for example, the following billiard tables (Fig. H3.2). From his 'experiments' he cast strong doubt about statistical mechanics. Briefly argue how Kelvin was correct (or not).

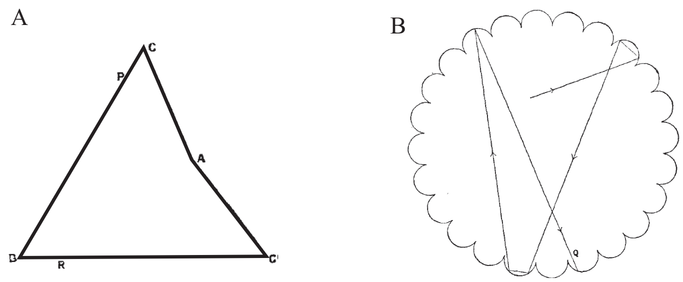

Figure H3.2: Left (A): Angles are A = 194°, B = 59°, C = C' = 53.5°. Right (B): The boundary consists of parts of circles whose centers are on a big circle. [Figs 5 and 6 of Kelvin's paper; Fig. 5 a bit cleaned]

Soln.
(A) is a typical polygonal billiard, so it cannot be ergodic for rational angle ratios. Kelvin's doubt is justified. However, actual molecular collisions (especially, spherical molecules) are quite unlikely to be modeled by such dynamics.
(B) This table satisfies the condition of Bunimovich's theorem, so it is strongly ergodic. Thus, Kelvin's 'simulation' was wrong.

In any case, Kelvin's idea was totally off the mark, but I appreciate his youthful drive (he was around 76; this may correspond to 85 or older now).

As you know quantization totally destroys ergodicity of the system. In this sense, you might say Kelvin was great, but to kill the equipartition of energy the only way is to deny even probability for energy distribution within classical mechanics, so what he did was the only possible choice and an easy one (an aging effect after all?)

3. An example of strange attractors simpler than Lorenz's is provided by the following set of equations: the Rössler system

$$\dot{x} = -y - z,\ \dot{y} = x + ay,\ \dot{z} = b + z(x - c). \tag{H3.4}$$

For $a = b = 0.2$ and $c = 14$ the attractor looks like Fig. H3.3.

You may enjoy the following videos:
`https://www.youtube.com/watch?v=o6w9CR7fk8s` bifurcation[4]

[4] There seems to be a 'period-doubling route' to chaos, but the 'doublings' is not infinite. You can see more details from "Rössler attractor" in Wikipedia. See also C LETELLIER and V MESSAGER, "INFLUENCES ON OTTO E. RÖSSLER'S EARLIEST PAPER ON CHAOS," International Journal of Bifurcation and Chaos 20 3585 (2010). 'Rössler' in Wikipedia depicts an eccentric person.

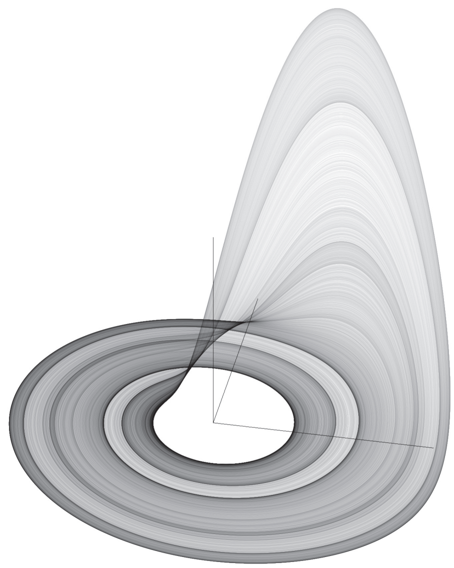

Figure H3.3: Rössler attractor [Wikipedia]

https://www.youtube.com/watch?v=BAWOk9LczSU Stereoplot of a trajectory

In this problem, let us make a discrete model (template) corresponding to the Rössler model, and clearly show a periodically perturbed relaxation oscillator can be chaotic.

The rule of the model is as follows:

The relaxation oscillator left alone has a strain energy accumulating speed $b > 1$, but

(i) if it is hit by an external periodic signal, its speed is reduced to b^{-1};

(ii) the speed is not affected by the hits if it is already b^{-1}, but

(iii) the speed is always reset to b when the oscillator returns to the lowest state.

The rule is very similar to that of Ito's earthquake model.

An example of the time evolution is in Fig. H3.4:

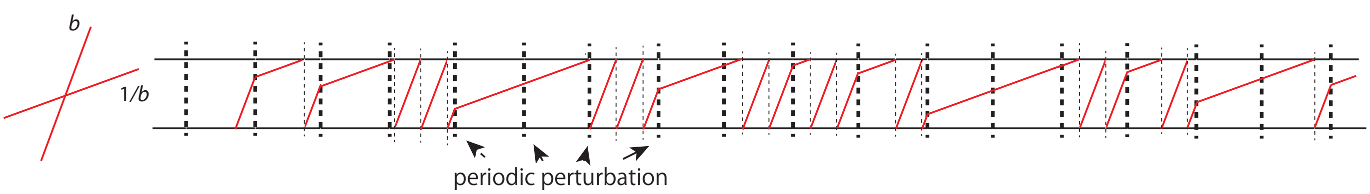

Figure H3.4: Periodically perturbed relaxation oscillator

We can plot the time evolution on the universal covering space as Fig. H3.5. As you see this is exactly the same as the Ito model.

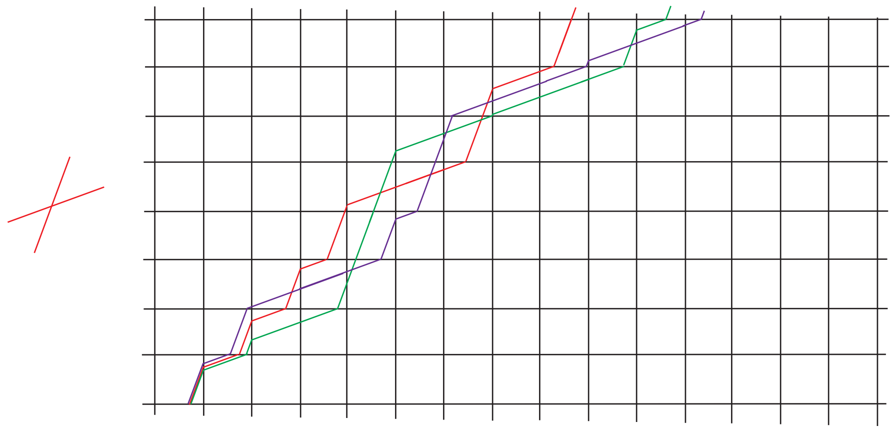

Figure H3.5: Time evolution on the universal covering space (three different initial conditions)

The rule may be visualized by using a torus (cut open; Fig. H3.6). In our case the external periodic hits need not be described, so we need one torus (square).

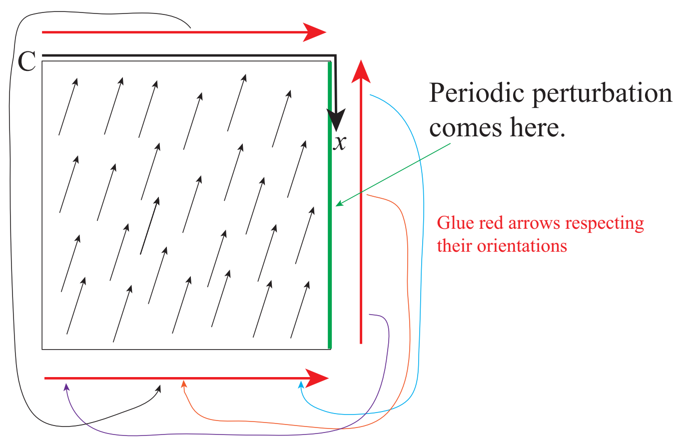

Figure H3.6: The rule; the square describes a flat 2-torus

Now we can start a topological acrobat as illustrated in Fig. H3.7.

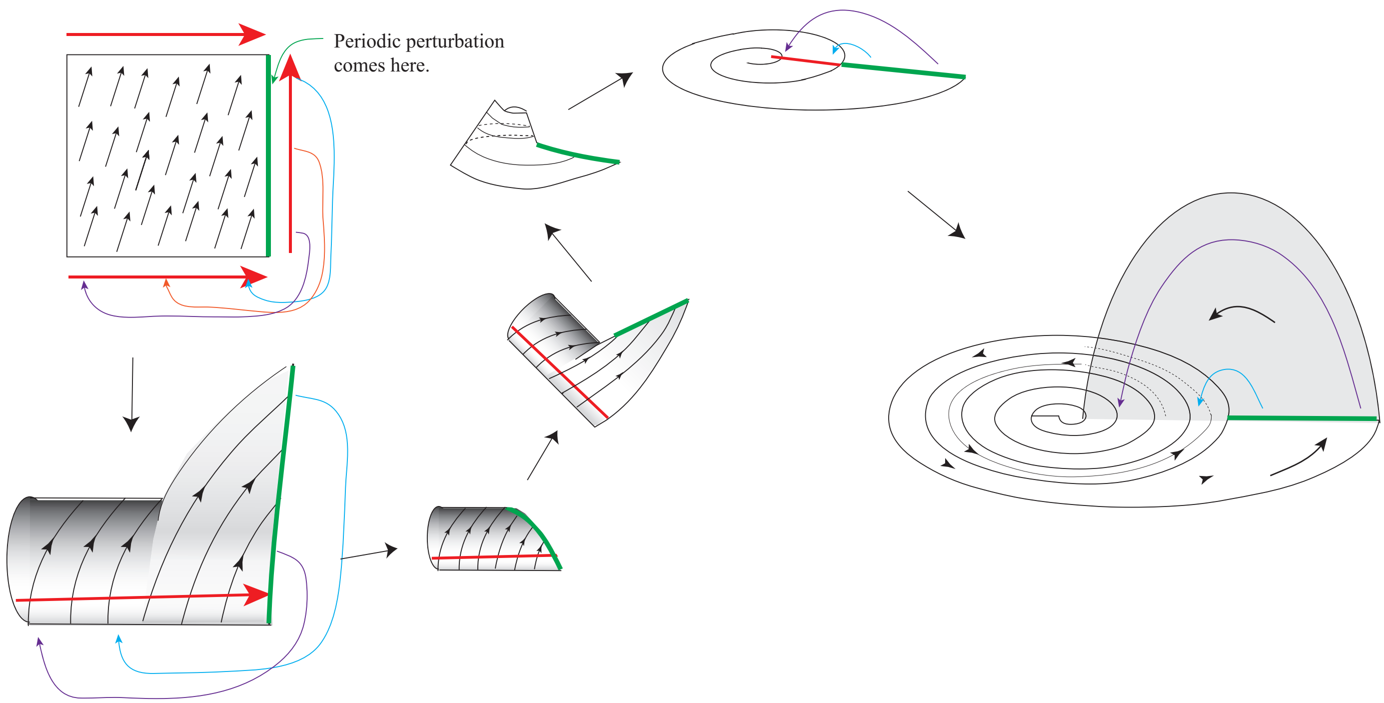

Figure H3.7: From ‘billiard’ to a template

3.1 To describe the dynamics of the system, we have only to record the crossing point of the upper and the right edges of the square in Fig. H3.6. We use the distance from C along these edges as the coordinate x to specify the crossing points. Make a map $F : [0,2] \to [0,2]$ (assume that the square is 1×1) from the n-th to the $(n+1)$th crossings, using this coordinate system.

Soln.
Note that the map should be piecewisely linear. $x = 1$ (the upper right corner) is mapped to 2 and then it is mapped to ‘a’ in the figure below. Near $x = a$ the map must have a slope 1 until you hit the corner $x = 1$. Also it must be symmetric wrt $x = 1$ as can be seen from the relation between the horizontal and vertical red arrows. The map must be continuous and piecewise-linear, so we have Fig. H3.8.

You could fold this system further as explained in Fig. 18.12 to obtain exactly the same map as the coupled oscillator example. Then, we have nothing to prove further (even 3.2 below is not needed)!

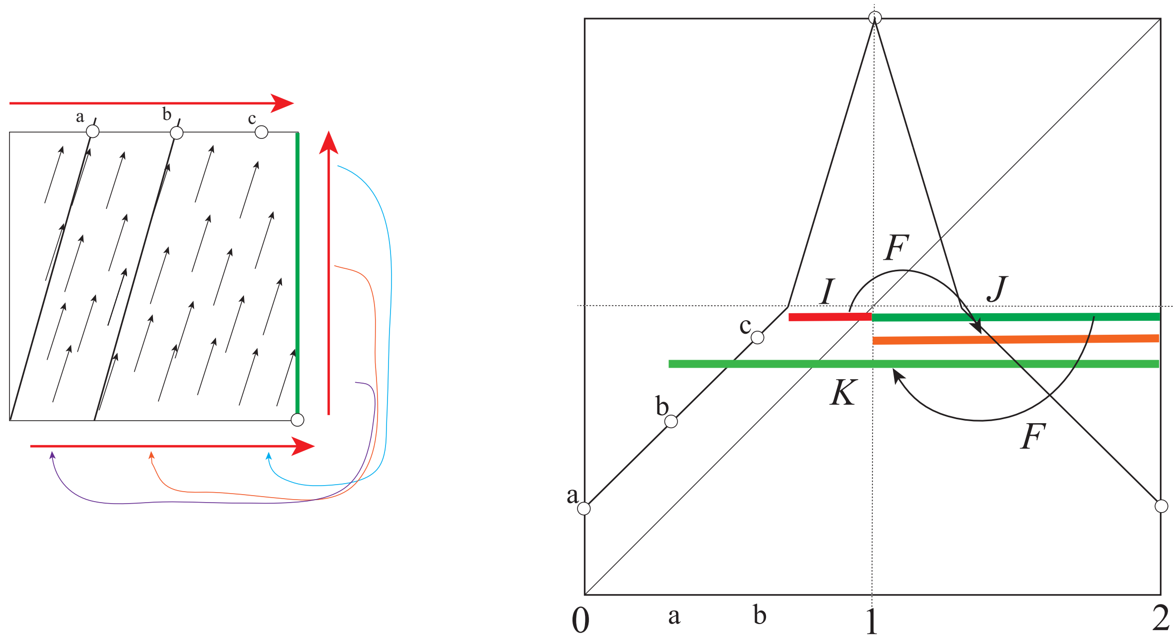

Figure H3.8:

3.2 Prove that this map exhibits chaos. [Thus, quite unambiguously we may conclude that the original continuous template system also exhibits chaos.]

Soln.

The easiest approach may be to check (5) of Theorem **22.15** (for the further folded map as mentioned above):

> There are two closed intervals J_1 and J_2 in the whole domain of the map that share at most one point such that $f^p(J_1) \cap f^q(J_2) \supset J_1 \cup J_2$ holds for some positive integers p and q.

In our case (look at Fig. H3.8) take I and J, which share one point $x = 1$. $F(I) = J$ and $F(J) = K$. Note that this holds quite generally and independently of the value of $b > 1$. Therefore, $F^2(I) \cap F(J) = K \supset I \cup J$. Thus, F can exhibit chaos. However, as noted in the text, this usually guarantees only the existence and does not guarantee its observability. However, in our case, the slopes of F are nowhere less than 1, so the observability of chaos is guaranteed (the invariant measure supporting this chaos is absolutely continuous).