

34.9 Exact partition function of 2-Ising model

I outline arguably the simplest (smartest) way to compute the 2D Ising partition function. The exposition is based on the first three sections of S Samuel, “The use of anticommuting variable integrals in statistical mechanics. I. The computation of partition functions,” J. Math. Phys. **21**, 2806 (1980).¹ I strongly urge you to read the first three sections.

To evaluate the partition function exactly is actually equivalent to mapping the system (at least asymptotically in the thermodynamic limit) to a noninteracting system, a system consisting of entities that do not interact with each other (that is, an ideal system like an ideal gas). The situation is exactly parallel with mechanics. We say a mechanical system is solvable when the system may be canonically transformed into a collection of free particles. Here, the 2D Ising model is mapped to a non-interacting fermion system.

The starting point is that any spin configuration on a 2-torus (= a square with periodic boundary conditions) can be identified with a set of non-overlapping closed curves (closed polygons) completely filling the space. It is intuitively obvious that *Bloch walls* (the boundary between the up and down spin domains) can completely specify a particular spin configuration. Bloch walls live on the dual lattice (Fig. 25.1).

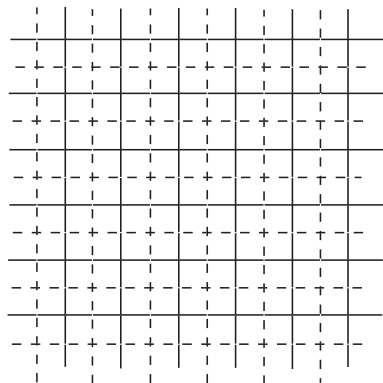


Fig. 25.1 Dual lattice. The dual lattice (dotted lattice) of a lattice (full line lattice) is the lattice made of the center points of the cells (or made of bonds evenly bisecting all the bonds) of the original lattice. The dual lattice of a square lattice is again a square lattice.

However, Bloch walls cannot completely fill up the space (cannot use up all the dual lattice points). Therefore, we add ‘tiny bubbles’ or ‘dots’ at the unused dual lattice points to use up all of them. This way, we can convert any spin configuration uniquely into a collection of the space filling and non-overlapping closed curves. Here, ‘non-overlapping’ means that two closed curves never share any edge (Bloch wall chunk). You can understand the conversion rule from the examples in Fig 25.2.

We introduce the partition function for polygons:

$$Z_{Ising}(J) = e^{2NJ} Z_{polygon}(z). \quad (0.0.1)$$

¹This article was recommended by Yonathan Shapir in 1981.

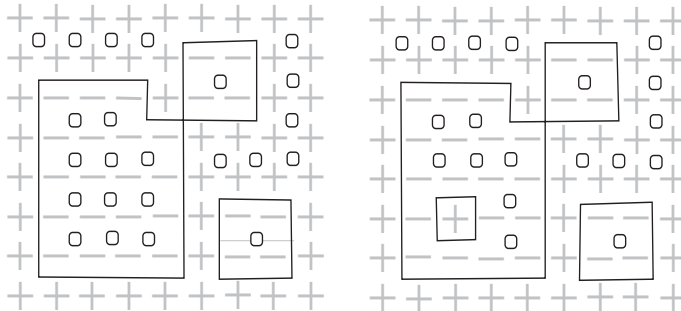


Fig 25.2 How to interpret spin configurations as space filling closed curves. Monomers (tiny closed curves) are needed to indicate clearly that there is no Bloch wall passing through the (dual) lattice point. Bloch walls can cross, but no two Bloch walls can share the same dual lattice bonds = no overlap of walls is allowed.

Here, $z = e^{-2J}$, and

$$Z_{polygon} = \sum^* z^L, \quad (0.0.2)$$

where L is the total length of the phase boundaries = Bloch walls (or the total number of bonds connecting up and down spins), and \sum^* means the summation over the ways to fill the lattice with closed non-overlapping polygons (including monomers, see Fig 25.2). Notice that $-2NJ$ is the total energy of the all up configuration. Thus, $Z_{polygon}$ is the partition function of the Ising model whose energy is measured from the all up configuration.

The next task is to encode the polygonal configurations into algebra. The key words are (i) space filling, and (ii) non-overlapping. (i) implies that all the dual lattice points must be used, and (ii) implies a dual lattice bond attached to a particular dual lattice point must not be used more than once. If we can use a variable η whose square is zero, $\eta^2 = 0$, we may describe ‘not more than once.’ Variables satisfying $\eta^2 = 0$ are inevitably *anticommuting*. Thus, we wish to encode the geometry in terms of an anticommuting algebra. If we can pick up the terms where all the variables show up, we can satisfy the space filling condition.

The relation between combinatorics and algebra may be illustrated by the following simple observation:

$$(1 + Ax)(1 + Bx) = 1 + (A + B)x + ABx^2. \quad (0.0.3)$$

Here, A and B correspond to parts of figures (configurations), and $+$ implies ‘or’ and product implies ‘and.’ The power of x implies the number of parts appearing in the configuration. Thus, the first term is without any parts (blank), the term with x implies the configurations with a single part A or B , and the term proportional to x^2 shows the configuration both A and B appear.

From now on we simply call dual lattice points and bonds lattice points and bonds. Let us introduce ‘hands’ associated with each lattice point p η_p^{h+} , η_p^{v+} , η_p^h and η_p^v (Fig 25.3) ($+$ implies the hands on the $+$ side of the coordinate; $h+$ = horizontal on the $+$ side, $v-$ = vertical on the negative side; etc).

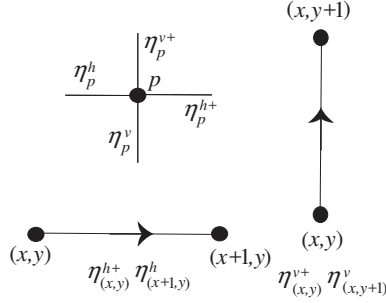


Fig. 25.3 ‘Hands’ associated with lattice point p that correspond to anti-commuting variables η_p^{h+} , η_p^{v+} , η_p^h and η_p^v . Also a horizontal Bloch wall chunk $\eta_{(x,y)}^{h+}\eta_{(x+1,y)}^h$ and a vertical Bloch wall chunk $\eta_{(x,y)}^{v+}\eta_{(x,y+1)}^v$ are shown. The arrows denote the order of the variables.

In terms of these variables a lattice bond (or a Bloch wall chunk or segment) may be described as $\eta_{(x,y)}^{v+}\eta_{(x,y+1)}^v$ or $\eta_{(x,y)}^{h+}\eta_{(x+1,y)}^h$ (Fig. 25.3).

For each configuration that fills the lattice with polygons (including monomers = points), we wish to use all the hand variables once and only once. If a horizontal Bloch wall passes through a lattice point p , then the horizontal hands are used up. The vertical hands are not used, but if you wish to prevent any vertical Bloch wall to go through p , a ‘preventer’ = monomer $\eta_p^{v+}\eta_p^v$ must be attached. If a Bloch wall has a corner at p , again to prevent it from touching with another Bloch wall corner, a ‘corner filler’ must be attached. These are illustrated in Fig 25.4. Thus, the Bloch wall crossing is possible only when four Bloch wall chunks join at one lattice point (thus multiple counting of crossing configurations is cleverly avoided).

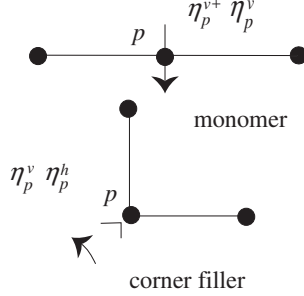


Fig. 25.4 Examples of ‘fillers’ or ‘preventors’ to isolate Bloch walls. The arrows denote the ordering of the variables. All the needed varieties are in Fig. 25.5.

Thus, to fill the lattice with polygons using up all the hands once and only once we need the following parts described in Fig. 25.5:

- (i) bloch walls (already shown in Fig 25.3),
- (ii) corner fillers,
- (iii) monomers.

For example, consider the following product:

$$\left[\prod_{p=1, \dots, 4} (1 + a_1 \eta_p^{h+} \eta_p^v) (1 + a_3 \eta_p^{v+} \eta_p^h) (1 + a_2 \eta_p^{v+} \eta_p^{h+}) (1 + a_4 \eta_p^v \eta_p^h) \right] \times \\ \times (1 + z \eta_1^{h+} \eta_2^h) (1 + z' \eta_1^{v+} \eta_4^v) (1 + z \eta_4^{h+} \eta_3^h) (1 + z' \eta_2^{v+} \eta_3^v) \times$$

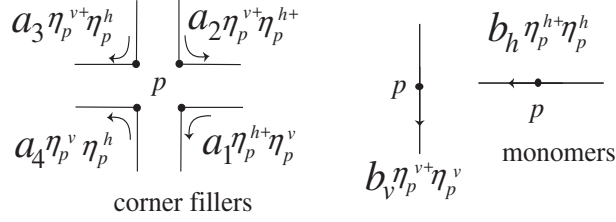


Fig. 25.5 ‘Filler’ for polygons. The arrows denote the ordering of the variables.

$$\times \left[\prod_{p=1, \dots, 4} (1 + b_h \eta_p^h \eta_p^{h+}) (1 + b_v \eta_p^v \eta_p^{v+}) \right] \quad (0.0.4)$$

If the product is expanded (respecting the anticommuting nature of the variables), the result consists of two kinds of terms; the terms containing all the hands associating with the 4 lattice points $1, \dots, 4$ once, and the terms that do not (notice that the terms with duplicated hands all vanish due to $\eta^2 = 0$). If we ignore the latter, we have only the following two types of terms (see Fig. 25.6):

$$a_1 a_2 a_3 a_4 z^2 z'^2 [\eta_1^v \eta_1^h \eta_2^{h+} \eta_2^v \eta_3^{v+} \eta_3^h \eta_4^{v+} \eta_4^h] [\eta_1^{v+} \eta_4^v \eta_1^{h+} \eta_2^h \eta_2^{v+} \eta_3^v \eta_4^{h+} \eta_3^h] \quad (0.0.5)$$

that corresponds to the simple plaquette and the terms that never contains any Bloch wall chunk. The latter may look like the right figure in Fig. 25.6. That is, at each (dual) lattice point is a pair of corner fillers or a pair of monomers.

$$a_1 a_3 a_2^2 a_4^2 b_h b_v [\eta_4^{h+} \eta_4^v \eta_4^{v+} \eta_4^h] [\eta_1^v \eta_1^h \eta_1^{v+} \eta_1^{h+}] [\eta_2^{h+} \eta_2^h \eta_2^{v+} \eta_2^v] [\eta_3^v \eta_3^h \eta_3^{v+} \eta_3^{h+}] \quad (0.0.6)$$

that corresponds to no Bloch wall cases (see Fig. 25.6). There is only one term of the form (0.0.5), but there are 81 terms analogous to (0.0.6), because at each lattice point (vertex) the mirror image arrangement of corner fillers is possible, and because also the monomer pairs can come in place of corner filler pairs.

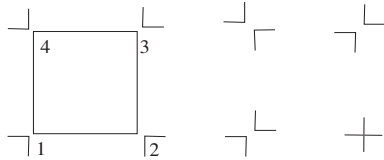


Fig. 25.6 Illustration of the terms in (0.0.5) (left) and (0.0.6) (right).

If we wish to describe the 2D Ising model, $|a_i| = 1$, and $|b_h| = |b_v| = 1$ are required, and

$$z = z' = e^{-2J} \quad (0.0.7)$$

for a symmetric model.

Let us briefly summarize the algebra and ‘calculus’ of anti-commuting variables η_α (*Grassmann variables*)

$$\eta_\alpha \eta_\beta + \eta_\beta \eta_\alpha = 0 \quad (0.0.8)$$

for any choice of α and β in the set of variables. In particular, $\eta_\alpha^2 = 0$. Thus,

$$e^{\eta_\alpha} = 1 + \eta_\alpha. \quad (0.0.9)$$

Therefore, if A is a monomial of Grassmann variables, then

$$e^A = 1 + A. \quad (0.0.10)$$

If A and B are sums of the even order monomials, then A and B commute, so

$$e^{A+B} = 1 + A + B + AB = e^A e^B. \quad (0.0.11)$$

Therefore, the product (0.0.4) can be rewritten (+ variables precede; within + variables (or no + variables) v precedes h in the current convention) as

$$\exp \left[\sum_{p=1, \dots, 4} (a_1 \eta_p^{h+} \eta_p^v + a_3 \eta_p^{v+} \eta_p^h + a_2 \eta_p^{v+} \eta_p^{h+} + a_4 \eta_p^v \eta_p^h + b_v \eta_p^{v+} \eta_p^v + b_h \eta_p^{h+} \eta_p^h) + z \eta_1^{h+} \eta_2^h + z' \eta_1^{v+} \eta_4^v + z \eta_4^{h+} \eta_3^h + z' \eta_2^{v+} \eta_3^v \right]. \quad (0.0.12)$$

We are sure that all the cluster configurations on the lattice appear once and only once in the expanded result. If we choose a_i and z as above, and if we declare that the values of the products containing all the Grassmann variables appearing in the system to be unity and other product values to be zero, then we can map the algebraic formula directly to the partition function. To invent such a map we have only to consider a linear map. A linear map from a set of monomials to a set of numbers compatible with the algebra of the variables may be interpreted as an integration.²

Let f be a function³ of n Grassmann variables $\eta_1 \cdots \eta_n$. It has the following general form:

$$f = a_0 + \sum_i a_{1i} \eta_i + \sum_{i < j} a_{2ij} \eta_i \eta_j + \cdots + a_{12 \dots n} \eta_1 \eta_2 \cdots \eta_n. \quad (0.0.13)$$

We introduce an integral of a function f of anticommuting variables as follows

$$\int d\eta_1 \cdots d\eta_n f = a_{12 \dots n}. \quad (0.0.14)$$

Here, the integration sign is determined by the ‘measure’ $d\eta_1 d\eta_2 \cdots d\eta_n$ specifying the ‘normal ordering’ of the variables. Notice that if we integrate (0.0.12), we obtain $(a_1 a_2 - b_h b_v + a_3 a_4)^2 + a_1 a_2 a_3 a_4 e^{4J}$.

The partition function for the polygons is given by

$$Z_{\text{polygon}}(z) = (-1)^N \int d\eta d\eta^+ e^A, \quad (0.0.15)$$

²You could call it a *trace*, anticommuting integral and write Tr , but the integration symbol is more convenient, because the normal order of the operators can be explicitly specified.

³We assume that f is analytic at the origin.

where $d\eta d\eta^+ = d\eta_1^h d\eta_1^v d\eta_1^{h+} d\eta_1^{v+} \dots$,⁴ N is the number of the sites and

$$A = A_{BW} + A_C + A_M, \quad (0.0.16)$$

$$A_{BW} = z \sum_{(x,y)} \left(\eta_{(x,y)}^{h+} \eta_{(x+1,y)}^h + \eta_{(x,y)}^{v+} \eta_{(x,y+1)}^v \right), \quad (0.0.17)$$

$$A_C = \sum_{(x,y)} \left(a_1 \eta_{(x,y)}^{h+} \eta_{(x,y)}^v + a_3 \eta_{(x,y)}^{v+} \eta_{(x,y)}^h + a_2 \eta_{(x,y)}^{v+} \eta_{(x,y)}^{h+} + a_4 \eta_{(x,y)}^v \eta_{(x,y)}^h \right), \quad (0.0.18)$$

$$A_M = \sum_{(x,y)} \left(b_h \eta_{(x,y)}^{h+} \eta_{(x,y)}^h + b_v \eta_{(x,y)}^{v+} \eta_{(x,y)}^v \right) \quad (0.0.19)$$

For the Ising model, from the above calculation for the single plaquette case, $a_1 a_2 a_3 a_4$ and $a_1 a_3 + a_2 a_4 - b_u b_h$ must be identical (and unity). Can we choose $a_i = b_h = b_v = +1$? We must check whether this choice (the sign choice) is OK.

The complication comes from the sign change when we ‘normal order’ the product. Corner fillers and monomers do not make any problem, because when their product is formed, the normal ordering does not change any sign. The problem may occur when two Bloch walls cross at (x, y) . We have the following product

$$\eta_{(x-1,y)}^{h+} \eta_{(x,y)}^h \eta_{(x,y)}^{h+} \eta_{(x+1,y)}^h \eta_{(x,y-1)}^{v+} \eta_{(x,y)}^v \eta_{(x,y)}^{v+} \eta_{(x,y+1)}^v, \quad (0.0.20)$$

that is, at (x, y) we have

$$\eta_{(x,y)}^h \eta_{(x,y)}^{h+} \eta_{(x,y)}^v \eta_{(x,y)}^{v+}. \quad (0.0.21)$$

Normal ordering flips the sign.

\Rightarrow vertex contributes -1

Let us put -1 to all vertices $a_i = b_h = b_v = -1$ and the overall $(-1)^N$

This way we have mapped the 2D Ising model to a free fermion model (it is free because the Hamiltonian is quadratic = Gaussian). As you know, the standard way to solve a free (Gaussian) model is Fourier transformation. Fourier transformation is a linear transformation, so even if the variables are anti-commuting, it is simply an organized fashion to construct convenient linear combinations of basic variables. Assuming that the lattice is $(2M + 1) \times (2M + 1)$, we introduce the following discrete Fourier transformation

$$\eta_{(x,y)}^* = \frac{1}{2M + 1} \sum_{s,t} a_{s,t}^* \exp \left(\frac{2\pi i x s}{2M + 1} + \frac{2\pi i y t}{2M + 1} \right), \quad (0.0.22)$$

or, you may regard this as the definition of $a_{s,t}^*$, where $*$ denotes $h, v+$, etc.

Introducing this into the definition of the Hamiltonian A , we obtain (the calculation is as usual; you have only to respect the order of the variables a_p^* which are anti-commuting)

$$A = \sum_{s,t} A_{s,t}, \quad (0.0.23)$$

⁴Notice that this ordering is exact opposite of the ordering in the product (0.0.12).

where

$$\begin{aligned}
A_{s,t} &= z a_{s,t}^{h+} a_{s,t}^h \exp\left(\frac{2\pi i s}{2M+1}\right) + z a_{s,t}^{v+} a_{s,t}^v \exp\left(\frac{2\pi i t}{2M+1}\right) \\
&\quad + a_1 a_{s,t}^{h+} a_{s,t}^v + a_3 a_{s,t}^{v+} a_{s,t}^h + a_2 a_{s,t}^{v+} a_{-s,-t}^{h+} + a_4 a_{s,t}^v a_{-s,-t}^h + b_h a_{s,t}^{h+} a_{s,t}^h + b_v a_{s,t}^{v+} a_{s,t}^v.
\end{aligned} \tag{0.0.24}$$

That is, (s, t) and $(-s, -t)$ couple, but there is no other couplings of variables.

We must perform the integration over η in (0.0.15), which is the integration over a . Since Fourier transformation is a unitary transformation, the Jacobian = 1. Thus, we can formally compute the integral as

$$Z_{\text{free fermion}} = \int da da^+ \prod_{s,t} A_{s,t} = \left[\prod_{s,t} L(s, t) \right]^{1/2}, \tag{0.0.25}$$

where

$$L\left(\frac{2\pi i s}{2M+1}, \frac{2\pi i t}{2M+1}\right) = \int da_{s,t} da_{-s,-t} A_{s,t} A_{-s,-t}, \tag{0.0.26}$$

where $da_{s,t} da_{-s,-t} = da_{s,t}^h da_{s,t}^v da_{s,t}^{h+} da_{s,t}^{v+} da_{-s,-t}^h da_{-s,-t}^v da_{-s,-t}^{h+} da_{-s,-t}^{v+}$. Here, we have used the evenness of L and taken the square root to avoid the double counting of (s, t) and $(-s, -t)$. The computation of (0.0.26) is straightforward. You need not be clever; simply write down all the terms and order the a variables in the normal order inherited from that of η . The result is

$$\begin{aligned}
&L\left(\frac{2\pi i s}{2M+1}, \frac{2\pi i t}{2M+1}\right) \\
&= h_s h_{-s} v_t v_{-t} - a_1 a_3 (h_s v_t + h_{-s} v_{-t}) - a_2 a_4 (h_s v_{-t} + h_{-s} v_t) + (a_1 a_3 + a_2 a_4)^2,
\end{aligned} \tag{0.0.27}$$

where

$$h_s = b_h - z \exp\left(\frac{2\pi i s}{2M+1}\right), \tag{0.0.28}$$

$$v_t = b_v - z \exp\left(\frac{2\pi i t}{2M+1}\right). \tag{0.0.29}$$

We take the thermodynamic limit $M \rightarrow \infty$, and the discrete Fourier transformation becomes the ordinary Fourier transformation

$$-\beta f_{\text{free fermion}} = \frac{1}{2} \int_{-\pi}^{\pi} \frac{dp_x}{2\pi} \int_{-\pi}^{\pi} \frac{dp_y}{2\pi} L(p_x, p_y). \tag{0.0.30}$$

Thus, the Ising model free energy is also obtained as

$$-\beta f_{\text{Ising}} = \log 2 + \frac{1}{2} \int_{-\pi}^{\pi} \frac{dp_x}{2\pi} \int_{-\pi}^{\pi} \frac{dp_y}{2\pi} \log[\cosh^2 2J + \sinh 2J(\cos p_x + \cos p_y)]. \tag{0.0.31}$$

The exact free energy was obtained by Onsager in 1944.⁵ (The elegant form given here is due to Schultz, Mattis and Lieb.) We can perform one integral with introducing $p_x \pm p_y$ as new integration variables:

$$-\beta f_{\text{Ising}} = -\log(2 \cosh 2J) - \frac{1}{2\pi} \int_0^\pi d\phi \log \frac{1}{2} \left(1 + \sqrt{1 - \kappa^2 \sin^2 \phi} \right), \quad (0.0.32)$$

where

$$\kappa = \frac{2}{\cosh 2J \coth 2J}. \quad (0.0.33)$$

From this, internal energy and specific heat can be computed by differentiation.⁶

The above method is of interest in its own right, because it is a typical technique to cast graphic problems into statistical mechanical problems. For example, with the same technique we can show that the non-self-tracing walks (the walks that never go through the same lattice bond twice) is in the same universality class of self-avoiding walk.

⁵L Onsager, Phys. Rev. **65**, 117 (1944).

⁶If you can demonstrate that there is a critical point, then its value may be exactly estimated. See R. P. Feynman, *Statistical Mechanics a set of lectures* (Benjamin, 1972) p139.