34.8 Perron-Frobenius theorem

Theorem [Perron and Frobenius]

Let A be a square matrix whose elements are all non-negative, and there is a positive integer n such that all the elements of A^n are positive. Then, there is a non-degenerate real positive eigenvalue λ such that
(i) $|\lambda_i| < \lambda$, where λ_i are eigenvalues of A other than λ,[1]
(ii) the elements of the eigenvector belonging to λ may be chosen all positive. □
This special real eigenvalue giving the spectral radius is called the *Perron-Frobenius eigenvalue*.

Proof of the Perron-Frobenius theorem[2]
Let us introduce the vectorial inequality notation: $\boldsymbol{x} > 0$ (≥ 0) implies that all the components of $\boldsymbol{x}$ are positive (non-negative). Also let us write $\boldsymbol{x} \geq (>) \boldsymbol{y}$ if $\boldsymbol{x} - \boldsymbol{y} \geq (>) 0$.
Let $\boldsymbol{x}$ be a vector such that $|\boldsymbol{x}| = 1$ and $\boldsymbol{x} \geq 0$. The largest ρ satisfying

$$A\boldsymbol{x} \geq \rho \boldsymbol{x} \tag{0.0.1}$$

is denoted by $\Lambda(\boldsymbol{x})$. The proof is divided into several steps.
(i) Since the set $U = \{\boldsymbol{x} \,|\, \boldsymbol{x} \geq 0, |\boldsymbol{x}| = 1\}$ is a compact set,[3] there is a vector $\boldsymbol{z} \in U$ that maximizes $\Lambda(\boldsymbol{x})$. Let us write $\lambda = \Lambda(\boldsymbol{z})$.
(ii) λ is an eigenvalue of A, and $\boldsymbol{z}$ belongs to its eigenspace: $A\boldsymbol{z} = \lambda \boldsymbol{z}$.
[Demo] Even if not, we have $\boldsymbol{w} = A\boldsymbol{z} - \lambda \boldsymbol{z} \geq 0$ (not equal to zero). Notice that for any vector $\boldsymbol{x} \geq 0$ but $\neq 0$ there is some positive integer m such that $A^m \boldsymbol{x} > 0$, so

$$A^m \boldsymbol{w} = AA^m \boldsymbol{z} - \lambda A^m \boldsymbol{z} > 0. \tag{0.0.2}$$

This implies $\Lambda(A^m \boldsymbol{z}) > \lambda$, but λ is the maximum of Λ, this is a contradiction. Therefore, $\boldsymbol{w} = 0$. That is, $\boldsymbol{z}$ is an eigenvector belonging to λ.
(iii) We may choose $\boldsymbol{z} > 0$.
[Demo] $z \geq 0$ and nonzero, so there is a positive integer m such that $A^m \boldsymbol{z} > 0$ but this is $\lambda^m \boldsymbol{z} > 0$, so actually $\boldsymbol{z} > 0$.
(iv) λ is the spectral radius of A.
[Demo] Suppose $A\boldsymbol{y} = \lambda' \boldsymbol{y}$. Let $\boldsymbol{q}$ be the vector whose components are absolute values of $\boldsymbol{y}$: $q_i = |y_i|$. Then, $A\boldsymbol{q} \geq |\lambda'| \boldsymbol{q}$. Therefore, $|\lambda'| \leq \lambda$.
(v) The absolute value of other eigenvalues are smaller than λ. That is, no eigenvalues other than λ is on the spectral circle.
[Demo] Suppose λ' is an eigenvalue on the spectral circle but is not real positive. Let $\boldsymbol{q}$ be the vector whose components are absolute values of an eigenvector belonging to λ'. Since $A\boldsymbol{q} \geq |\lambda'| \boldsymbol{q} = \lambda \boldsymbol{q}$, actually we must have $A\boldsymbol{q} = \lambda \boldsymbol{q}$. That is, the absolute value of each component of the vector $A^m \boldsymbol{y} = \lambda'^m \boldsymbol{y}$ coincides with the corresponding component of $A^m \boldsymbol{q}$. This implies

$$\left| \sum_j (A^m)_{ij} y_j \right| = \sum_j (A^m)_{ij} |y_j| = \sum_j |(A^m)_{ij} y_j|. \tag{0.0.3}$$

All the components of A^m are real positive, so all the arguments of y_j are identical.[4] Hence, $\lambda' = \lambda$.

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[1]That is, λ gives the spectral radius of A.

[2]A standard reference may be E. Seneta, *Non-negative matrices and Markov chains* (Springer, 1980). The proof here is an eclectic version due to many sources, including N. Iwahori, *Graphs and Stochastic Matrices* (Sangyo-tosho, 1974).

[3]The domain of λ is a sphere of some dimension restricted to the nonnegative coordinate sector. The existence uses the fixed point theorem.

[4]$a, b \neq 0$ and $|a + b| = |a| + |b|$ imply the real positivity of a/b.

(vi) λ is non-degenerate.
[Demo] Suppose otherwise. Then, there is a vector $\boldsymbol{z}'$ that is not proportional to $\boldsymbol{z}$ but still $A\boldsymbol{z}' = \lambda\boldsymbol{z}'$. Here, A is a real matrix and λ is real, we may choose $\boldsymbol{z}'$ to be real. Since $\boldsymbol{z}$ and $\boldsymbol{z}'$ are not parallel, we may choose α appropriately so that $\boldsymbol{v} = \boldsymbol{z} + \alpha\boldsymbol{z}' \geq 0$ but has a zero component. This is contradictory to (iii).