33.1 Simon's theorem

At the second order phase transition point, the correlation length diverges. What happens to the correlation function? (**??**) implies that the correlation must decay slower than any exponential decay. For some $c > 0$ the decay like $e^{-c\sqrt{r}}$ can be imagined, but such a stretched exponential decay is impossible for equilibrium spatial correlation functions. For example, when ferromagnetic ordering occurs, it can be proved[1] that

$$\langle \phi(0)\phi(\boldsymbol{r})\rangle \le \sum_{\boldsymbol{s}\in B} \langle \phi(0)\phi(\boldsymbol{s})\rangle \langle \phi(\boldsymbol{s})\phi(\boldsymbol{r})\rangle. \tag{0.0.1}$$

Here, B can be any set of lattice points such that removal of all the points in B destroys all the paths along the lattice bonds connecting the origin 0 and $\boldsymbol{r}$. For example, we may choose it to be a spherical shell centered at the origin with an appropriate thickness.[2] If the system is translationally symmetric, then from (0.0.1) we get

$$\langle \phi(\boldsymbol{s})\phi(\boldsymbol{r})\rangle = \langle \phi(0)\phi(\boldsymbol{r}-\boldsymbol{s})\rangle \le \sum_{\boldsymbol{s}_1\in B} \langle \phi(0)\phi(\boldsymbol{s}_1)\rangle \langle \phi(\boldsymbol{s}_1)\phi(\boldsymbol{r}-\boldsymbol{s})\rangle \tag{0.0.2}$$

All the moments of spins are positive.[3] Therefore, introducing the relation (0.0.2) into (0.0.1), we obtain

$$\langle \phi(0)\phi(\boldsymbol{r})\rangle \le \sum_{\boldsymbol{s}\in B} \langle \phi(0)\phi(\boldsymbol{s})\rangle \left(\sum_{\boldsymbol{s}_1\in B} \langle \phi(0)\phi(\boldsymbol{s}_1)\rangle \langle \phi(\boldsymbol{s}_1)\phi(\boldsymbol{r}-\boldsymbol{s})\rangle \right). \tag{0.0.3}$$

To be sure let us reiterate this procedure once more:

$$\langle \phi(0)\phi(\boldsymbol{r})\rangle \le \sum_{\boldsymbol{s}\in B} \langle \phi(0)\phi(\boldsymbol{s})\rangle \left\{ \sum_{\boldsymbol{s}_1\in B} \langle \phi(0)\phi(\boldsymbol{s}_1)\rangle \left(\sum_{\boldsymbol{s}_2\in B} \langle \phi(0)\phi(\boldsymbol{s}_2)\rangle \langle \phi(\boldsymbol{s}_2)\phi(\boldsymbol{r}-\boldsymbol{s}-\boldsymbol{s}_1)\rangle \right) \right\}. \tag{0.0.4}$$

If $\boldsymbol{r}$ is sufficiently far away from the origin and the above procedure may be repeated q times, then replacing the term $\langle \phi(\boldsymbol{s}_q)\phi(\boldsymbol{r}-\boldsymbol{s}-\boldsymbol{s}_1-\cdots-\boldsymbol{s}_{q-1})\rangle$ that is expected to appear after these iterations with its maximum value C, we obtain

$$\langle \phi(0)\phi(\boldsymbol{r})\rangle \le \left[\sum_{\boldsymbol{s}\in B} \langle \phi(0)\phi(\boldsymbol{s})\rangle \right]^q C. \tag{0.0.5}$$

Here, q may be chosen to be a number proportional to $|\boldsymbol{r}|$. If we can make $\sum_{\boldsymbol{s}\in B}\langle \phi(0)\phi(\boldsymbol{s})\rangle < 1$, the correlation function decays exponentially. That is, the summability of the correlation function on a large sphere is the key for exponential decay. If the correlation function behaves as $r^{-\mu}$ ($d-1<\mu$), the decay must be exponential. On the other hand, as can be seen from the decay of the fundamental solution to the Laplace equation, even without any fluctuation,

[1] For ferromagnetic Ising models, see B. Simon, "Correlation Inequalities and the decay of correlations in ferromagnets," Commun. Math. Phys., **77**, 111 (1980).

[2] For d-(hyper)cubic lattice, its thickness can be slightly larger than $\sqrt{d}\,\times$ lattice spacing.

[3] This is one of *Griffiths' inequalities* (the first inequality): for any finitely many positive integers a_i $\langle \prod_i s_i^{a_i}\rangle \ge 0$. A simple proof may be found in J. Glimm and A. Jaffe, *Quantum Physics, a functional integral point of view*, Second Edition (Springer, 1987) Sect. 4.1.

correlation decays as $1/r^{d-2}$.

If there are fluctuations, intuitively speaking at least for the ferromagnetic case, the correlation is expected to decay faster than without fluctuations. Therefore, when ξ diverges, the correlation function must decay algebraically; if we write this algebraic decay as $r^{-\mu}$, μ cannot be larger than $d-1$, and cannot be smaller than $d-2$. Therefore, we may conclude that**r**

$$\langle \phi(0)\phi(\boldsymbol{r})\rangle \sim \frac{1}{r^{d-2+\eta}} \tag{0.0.6}$$

is the general expression for the order parameter correlation function at the critical point. Here, η ($\in (0,1]$) is one of the critical indices we will encounter in the next section.

The above argument suggests that if there is a long range interaction, fluctuation effects can be contained. We already know such an example: the 1D Kac model. Even in 1-space this long-range interaction model exhibits gas-liquid phase transition.[4]

What happens if we introduce a small number of (fraction of) long range interactions into a short-ranged lattice system? This is a story of systems with the so-called 'small world' interaction network.[5] Let us introduce such long range interaction with probability p ($\ll 1$). The average distance from an arbitrary lattice point to one of such long-range interacting points (= airports) is $\ell \sim p^{-1/d}$. We compare this with the domain size = correlation length ξ, which is the distance over which spins can 'communicate.' This length generally grows as temperature decreases. The ordering would occur if ξ reaches ℓ, because then the ordered patch can have access to the 'big world.' As we will learn, in 1-space[6]

$$\xi \sim e^{2J/k_BT}, \tag{0.0.7}$$

so $T_c \sim 1/|\log p|$; now phase transition can occur at a positive temperature. Also as we will learn later $\xi \sim (T-T_c)^{-\nu}$ in $D(>1)$-space, so $T_c(p) - T_c(0) \sim p^{1/d\nu}$.

[4]However, no local order can be stabilized by a long range interaction, so no crystal formation is possible.

[5]see D. J. Watts, *Small Worlds: The Dynamics of Networks between Order and Randomness* (Princeton Studies in Complexity, 2003).

[6]The probability of introducing one up-down boundary is e^{-2J/k_BT}, so the spacing of 'defects' should be the reciprocal of this.