

The Mermin-Wagner theorem tells us that there is no spontaneous magnetization in 2-space for spin dimension larger than or equal to  $n = 2$ . Here, let us study the  $n = 2$  case: the XY-model.

### 0.0.1 XY-model Hamiltonian

At each lattice point is a unit vector  $\mathbf{s}_i$  pointing in any direction in a plane. There is a nearest neighbor interaction given by  $-J\mathbf{s}_i \cdot \mathbf{s}_j$ . The the total Hamiltonian of the system is

$$H = -J \sum_{\langle i,j \rangle} \mathbf{s}_i \cdot \mathbf{s}_j = -J \sum_{\langle i,j \rangle} \cos[\theta(i) - \theta(j)]. \quad (0.0.1)$$

Here,  $\theta(i)$  denotes the angle of the direction of  $\mathbf{s}_i$  measured from (say) the  $x$ -axis. If we assume that the lattice spacing  $a$  is small and the spin directions do not change too rapidly from point to point, then  $\theta(i) - \theta(j) \sim a\nabla\theta$ , so a continuous version of (0.0.1) reads

$$H = -\frac{1}{2}J \int d^2\mathbf{r} (\nabla\theta)^2. \quad (0.0.2)$$

(Here,  $J$  is not the same  $J$  in (0.0.1), but  $Ja^2$ , but for the time being we ignore  $a$ .)

**Warning:** Since  $\theta$  and  $\theta + 2n\pi$  ( $n \in \mathbb{Z}$ ) are indistinguishable,  $\theta$  is not a potential, and  $\nabla\theta$  is not a gradient of a function.

You cannot scale  $\theta$ . For the ordinary  $\phi^4$  model, we scale the field  $\phi$  to set  $J$  to be unity, but scaling  $\theta$  changes the indistinguishability of  $\theta + 2n\pi$ , we cannot change  $J$  by scaling  $\theta$ . Thus,  $J$  or  $K = \beta J$  appears in observables.

### 0.0.2 Very high temperature phase

Let us study the nearest-neighbor correlation

$$C = \langle \cos(\theta_1 - \theta_0) \rangle = \langle \cos \delta\theta \rangle = \langle e^{i\delta\theta} \rangle \quad (0.0.3)$$

The last equality follows from the invariance of the Hamiltonian under spin slip  $\theta \rightarrow -\theta$ :  $\langle e^{\delta\theta} \rangle = \langle e^{-\delta\theta} \rangle$ . The standard approach to study the high temperature behavior is the high-temperature expansion, but here we do not proceed systematically.

$$C = \int_0^{2\pi} d\delta\theta e^{i\delta\theta} e^{-\beta\mathcal{H}} / Z \quad (0.0.4)$$

Since  $K = \beta J$  is very small, we can expand

$$e^{-\beta\mathcal{H}} = 1 + K \cos \delta\theta + \sum' [K \cos(\theta(0) - \theta') + K \cos(\theta(1) - \theta'')] + \dots + O[K^2], \quad (0.0.5)$$

where the term with  $\sum'$  denotes the interaction between the pair we look at and its neighbor spin interactions. Obviously,  $Z = 2\pi + O[K^2]$ . Thus

$$C = \int_0^{2\pi} d\delta\theta e^{i\delta\theta} (1 + K \cos(\delta\theta) + \Sigma' + \dots + O[K^2]) \sim \frac{1}{2\pi} \int_0^{2\pi} d\delta\theta e^{i\delta\theta} \frac{K}{2} e^{-i\delta\theta} = \beta J / 2. \quad (0.0.6)$$

Here,  $\cos(\delta\theta) = (e^{i\delta\theta} + e^{-i\delta\theta})/2$  is used (the integral of  $e^{2i\delta\theta}$  vanishes). The contribution of  $\Sigma'$  is of order  $K^2$ , because  $[\alpha \cos(\theta(0) - \theta')] \sim K$  (according to the just finished calculation)

so we ignore all other terms. Thus, we may guess (if you do not like a mere guess read the fine letters just below)

$$\langle \cos(\theta_r - \theta_0) \rangle \sim \left(\frac{K}{2}\right)^r = \exp(-r \log(2/K)) \quad (0.0.7)$$

That is, the spin-spin correlation function decays exponentially. The correlation length reads

$$\xi = 1/\log(2/K) \sim 1/\log T. \quad (0.0.8)$$

Notice that  $\cos(\theta_0 - \theta_2) = \cos(\theta_0 - \theta_1)\cos(\theta_1 - \theta_2)$  – sine terms, but the average of the sign terms vanish. Thus,

$$\langle \cos(\theta_r - \theta_0) \rangle \sim \sum_{\text{path}} \left(\frac{K}{2}\right)^r, \quad (0.0.9)$$

where the summation over ‘path’ means the sum over all the path connecting 0 and  $r$  spins. The shortest is length  $r$ . The number of paths of length  $r + m$  ( $m \geq 1$ ) connecting spin 0 and  $r$  cannot be larger than  $4^m$ , so their contribution cannot be larger than  $K^r \times (2K)^m$  and the total sum cannot be larger than  $2K(1 - 2K) \times K^r$ . We can choose  $K$  sufficiently small to ignore these contributions.

### 0.0.3 k-space expression of the Hamiltonian

We introduce the Fourier transformation of the field  $\theta$ :

$$\theta(\mathbf{k}) = \int d\mathbf{r} \theta(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}}, \quad (0.0.10)$$

$$\theta(\mathbf{r}) = \frac{1}{4\pi^2} \int d\mathbf{k} \theta(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}}. \quad (0.0.11)$$

Introducing the  $k$ -space expression into (0.0.2)

$$\begin{aligned} \int d\mathbf{r} (\nabla\theta)^2 &= \int d\mathbf{r} \frac{1}{4\pi^2} \int d\mathbf{k} (-i\mathbf{k})\theta(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}} \cdot \frac{1}{4\pi^2} \int d\mathbf{k}' (-i\mathbf{k}')\theta(\mathbf{k}') e^{-i\mathbf{k}'\cdot\mathbf{r}} \\ &= \frac{1}{4\pi^2} \int d\mathbf{k} k^2 \theta(\mathbf{k}) \theta(-\mathbf{k}) \end{aligned} \quad (0.0.12)$$

Thus,

$$H = -\frac{J}{2(2\pi)^2} \int d\mathbf{k} k^2 \theta(\mathbf{k}) \theta(-\mathbf{k}). \quad (0.0.13)$$

### 0.0.4 k-space expression of correlation

We get straightforwardly

$$\langle \theta(\mathbf{r})\theta(0) \rangle = \frac{1}{(4\pi^2)^2} \int d\mathbf{k} \int d\mathbf{k}' \langle \theta(\mathbf{k})\theta(\mathbf{k}') \rangle e^{-i\mathbf{k}\cdot\mathbf{r}}. \quad (0.0.14)$$

### 0.0.5 Complex Gaussian

Let the Gaussian distribution of the field  $\phi$  is given by

$$\exp \left[ -\frac{1}{2} \int d\mathbf{r} d\mathbf{r}' \phi(\mathbf{r}) \phi(\mathbf{r}') A(\mathbf{r} - \mathbf{r}') \right] \quad (0.0.15)$$

Introducing the Fourier expression of  $\phi$  the quantity inside [ ] reads

$$-\frac{1}{2} \int d\mathbf{r} d\mathbf{r}' \frac{1}{4\pi^2} \int d\mathbf{k} \phi(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}} \frac{1}{4\pi^2} \int d\mathbf{k}' \phi(\mathbf{k}') e^{-i\mathbf{k}'\cdot\mathbf{r}'} A(\mathbf{r} - \mathbf{r}') \quad (0.0.16)$$

$$= -\frac{1}{2(4\pi^2)^2} \int d\mathbf{k} d\mathbf{k}' \phi(\mathbf{k}) \phi(\mathbf{k}') A(\mathbf{k}) \delta(\mathbf{k} + \mathbf{k}') \quad (0.0.17)$$

where

$$\int d\mathbf{r} d\mathbf{r}' e^{-i\mathbf{k}\cdot\mathbf{r} - i\mathbf{k}'\cdot(\mathbf{r}')} A(\mathbf{r} - \mathbf{r}') \quad (0.0.18)$$

$$= \int d\mathbf{r} d\mathbf{s} e^{-i\mathbf{k}\cdot\mathbf{r} - i\mathbf{k}'\cdot(\mathbf{r}-\mathbf{s})} A(\mathbf{s}) \quad (0.0.19)$$

$$= 4\pi^2 \delta(\mathbf{k} + \mathbf{k}') \int d\mathbf{s} e^{-i\mathbf{k}\cdot\mathbf{s}} A(\mathbf{s}) = 4\pi^2 A(\mathbf{k}) \delta(\mathbf{k} + \mathbf{k}'). \quad (0.0.20)$$

This implies

$$\langle \phi(\mathbf{k}) \phi(\mathbf{k}') \rangle = \frac{4\pi^2}{A(\mathbf{k})} \delta(\mathbf{k} + \mathbf{k}'). \quad (0.0.21)$$

### 0.0.6 Very low temperature correlation

At very low temperatures,  $\theta$  changes mostly smoothly, so it is quite unlikely that locally  $\theta$  changes rapidly. Therefore, we may ignore that  $\theta$  is an angle and regard it as the ordinary function (called ‘spin-wave approximation’). Then, we may use (0.0.2) disregarding the ‘periodicity.’ Then,  $\theta$  is a Gaussian field, obeying (see (0.0.13))

$$\sim \exp \left[ -\frac{K}{2(2\pi)^2} \int d\mathbf{k} k^2 \theta(\mathbf{k}) \theta(-\mathbf{k}) \right]. \quad (0.0.22)$$

Thanks to the Gaussianness of  $\theta$  (recall that cumulants higher than the second order vanish)

$$\langle \cos(\theta(\mathbf{r}) - \theta(0)) \rangle = \langle e^{i(\theta(\mathbf{r}) - \theta(0))} \rangle = \exp \left( -\frac{1}{2} \langle (\theta(\mathbf{r}) - \theta(0))^2 \rangle \right) = e^{-[g(0) - g(\mathbf{r})]}, \quad (0.0.23)$$

where

$$g(\mathbf{r}) = \langle \theta(\mathbf{r}) \theta(0) \rangle. \quad (0.0.24)$$

This can be computed as (0.0.14). Looking at the result of 0.0.5 we find

$$\langle \theta(\mathbf{k}) \theta(\mathbf{k}') \rangle = \frac{4\pi^2}{K k^2} \delta(\mathbf{k} + \mathbf{k}'). \quad (0.0.25)$$

Hence,

$$g(\mathbf{r}) = \frac{1}{(4\pi^2)^2} \int d\mathbf{k} \int d\mathbf{k}' \frac{4\pi^2}{Kk^2} \delta(\mathbf{k} + \mathbf{k}') e^{-i\mathbf{k}\cdot\mathbf{r}} = \frac{1}{4\pi^2 K} \int d\mathbf{k} \frac{1}{k^2} e^{-i\mathbf{k}\cdot\mathbf{r}} \quad (0.0.26)$$

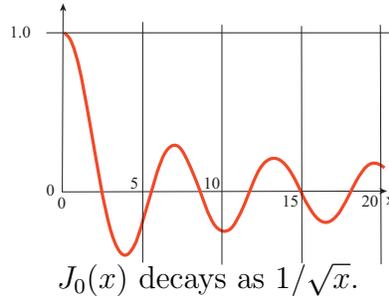
We need

$$g(0) - g(\mathbf{r}) = \frac{1}{4\pi^2 K} \int d\mathbf{k} \frac{1}{k^2} (1 - e^{-i\mathbf{k}\cdot\mathbf{r}}). \quad (0.0.27)$$

To calculate this, we integrate over the angle variable first. We use (whenever  $\cos \theta$  appears in the exponent think of Bessel functions!)

$$\frac{1}{2\pi} \int d\theta e^{ikr\cos\theta} = J_0(kr) \quad (0.0.28)$$

which behaves as in the following figure.



Thus, after integration over the angle (note  $d^2\mathbf{k} = d\theta k dk$ )

$$g(0) - g(\mathbf{r}) = \frac{1}{2\pi K} \int dk \frac{1}{k} (1 - J_0(kr)). \quad (0.0.29)$$

This integral is not finite (logarithmically divergent), so an atomic scale cutoff  $a$  is needed. Then,

$$g(0) - g(\mathbf{r}) = \frac{1}{2\pi K} \int^{1/a} dk \frac{1}{k} (1 - J_0(kr)) = \frac{1}{2\pi K} \int^{r/a} dy \frac{1}{y} (1 - J_0(y)). \quad (0.0.30)$$

Close to zero  $J_0$  is holomorphic, so the integral converges near the lower end. Near the higher end, if  $r/a$  is very large, we may ignore  $J_0$ . therefore

$$g(0) - g(\mathbf{r}) = \frac{1}{2\pi K} \int^{1/a} dk \frac{1}{k} (1 - J_0(kr)) = \frac{1}{2\pi K} \log(r/a). \quad (0.0.31)$$

We have obtained

$$\langle \cos(\theta(\mathbf{r}) - \theta(0)) \rangle = \exp \left[ -\frac{1}{2\pi K} \log(r/a) \right] = \left( \frac{a}{r} \right)^{1/2\pi K}. \quad (0.0.32)$$

We say there is a *quasi long-range order*.

### 0.0.7 There must be a phase transition

We have seen that at higher temperatures the correlation decays exponentially, and at very low temperatures, there is a quasi long-range order. Thus, there is a qualitative change at least at one temperature.

### 0.0.8 Topological excitations

We assumed that the angle field is simply meandering in the ‘spin wave approximation.’ This is equivalent to assuming that the vector field is irrotational. It is known that any vector field in 2-space can be decomposed into the sum of the irrotational  $\nabla\psi$  and rotational fields. Therefore, symbolically the *nabla*theta we have so far used into two parts

$$\nabla\theta \rightarrow \nabla\psi + \nabla\theta, \quad (0.0.33)$$

and  $\nabla\theta$  as its rotational portion. For example,  $\nabla\theta$  with nonzero curl is obtained from the vector field of unit vectors in Fig. 0.0.1.

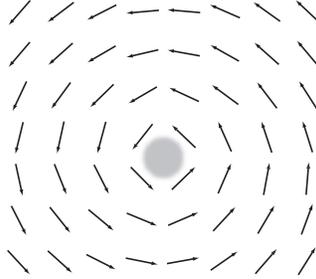


Figure 0.0.1:

Notice that the field in the figure gives  $\mathbf{s}(x, y) = (-y/r^2, x/r^2) = \nabla\theta(x, y)$ , so Green’s theorem tells us

$$\int_{\partial D} \mathbf{s} \cdot d\boldsymbol{\ell} = \int_D \Delta\theta d^2\mathbf{r} \quad (0.0.34)$$

but

$$\int_{\partial D} \mathbf{s} \cdot d\boldsymbol{\ell} = \int_{\partial D} (-\sin\theta/r, \cos\theta/r) \cdot (-r\sin\theta, r\cos\theta)d\theta = -\int_{\partial D} d\theta = -2\pi \quad (0.0.35)$$

$\Delta\theta = 0$  everywhere except at the core of the vortex, so we can identify a topological charge of +1 at the core.

### 0.0.9 Topological charges

There are various vortices characterized by the charge  $q$  computed as (0.0.35)

$$\Delta\theta = -2\pi q \quad (0.0.36)$$

Even if the charges are identical, the spin configurations look drastically different, so do not be fooled by the patterns, Simply imagine a unit circle around the core, and count how many time the spin vectors rotate when you go around the circle positively once.

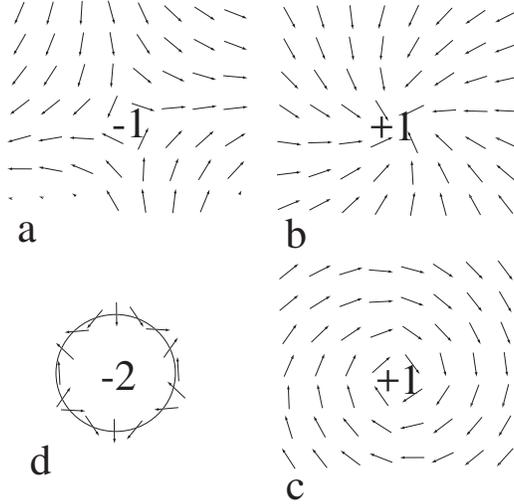


Figure 0.0.2:

The Poisson equation for  $\theta$  suggests that  $\nabla\theta = \mathbf{s}$  introduced above may be understood as an analogue of electric field. Let us compute the energy (0.0.2) of this field configuration. Note that  $|\nabla\theta|^2 = |\mathbf{s}|^2 = 1/r^2$ , so the energy increases due to the presence of the vortex is given by

$$\delta E = \frac{1}{2}J \int_a^R \frac{1}{r^2} d^2\mathbf{r} = \pi J \log \frac{R}{a}. \quad (0.0.37)$$

Here,  $a$  is the cut off (the core size of the vortex, or the lattice spacing if the model is defined on a lattice) and  $R$  is the system size. This is just the 2-Coulomb potential energy.

### 0.0.10 Topological charge-electric charge analogy

We can say that the field created by a single vortex of charge  $q$  makes an ‘electric field’  $\mathbf{E} = \nabla\theta$  such as

$$\mathbf{E} = (-\sin\theta, \cos\theta) \frac{1}{r}. \quad (0.0.38)$$

Suppose we have two vortices at  $\mathbf{r}$  and at  $\mathbf{r}'$  whose charges are  $q$  and  $q'$ , respectively. Let  $\mathbf{E}$  be the ‘electric field’ created by charge  $q$  and  $\mathbf{E}'$  by charge  $q'$ . Then, the energy  $\delta E$  needed to create this pair is obtained as

$$\delta E = \frac{1}{2}J \int (\mathbf{E}(\mathbf{x}) + \mathbf{E}'(\mathbf{x}))^2 d^2\mathbf{x}. \quad (0.0.39)$$

The interaction energy (‘Coulomb energy’) is given by the cross term:

$$J \int \mathbf{E}(\mathbf{x}) \cdot \mathbf{E}'(\mathbf{x}) d^2\mathbf{x}. \quad (0.0.40)$$

Our ‘electric field’ is different from the ordinary electric field due to electrical charges, but the only difference is that the direction of the field is rotated by 90° degrees with respect to

the corresponding true electric field. Therefore, the scalar product in (0.0.40) is exactly the same as the ordinary Coulomb interaction in 2-space. Therefore

$$\delta E_{interaction} = -2\pi q q' J \log |\mathbf{r} - \mathbf{r}'|. \quad (0.0.41)$$

The contributions from  $\mathbf{E}^2$  and  $\mathbf{E}'^2$  give the self-energies of the charges. With a finite core size  $a$ , these become finite, and may be understood as the core energy. Let us write it as  $\mu q^2$ , because the field intensity is proportional to  $q$ . Thus, if we have many charges (vortices), their contribution may be written as

$$\delta E = -2\pi J \sum_{\langle i,j \rangle} q_i q_j \log \frac{|\mathbf{r}_i - \mathbf{r}_j|}{a} + \mu \sum q_i^2, \quad (0.0.42)$$

where  $\mathbf{r}_i$  is the position of charge  $q_i$ . Here, the cutoff  $a$  is explicitly written. We assume that the total charge is zero to avoid a large system dependent term proportional to  $\log R$  as seen from the single vortex calculation above.

There is some complication due to the definition of the charge, so let us look at a 3D analogue: Usually

$$\Delta \phi = -\frac{q}{\epsilon_0} \delta(\mathbf{r}) \Rightarrow \phi = \frac{q}{4\pi \epsilon_0 r} \quad (0.0.43)$$

and the Coulomb interaction between two charges reads

$$V = \frac{q q'}{4\pi \epsilon_0 r} \quad (0.0.44)$$

Now, let us replace the charge with a more natural ‘topological charge (hedge hog charge)  $Q$ :

$$\Delta \phi = -4\pi Q \quad (0.0.45)$$

Therefore,  $Q = q/4\pi \epsilon_0$ . Therefore, the intereaction energy reads

$$V = \frac{(4\pi \epsilon_0)^2 Q Q'}{4\pi \epsilon_0 r} = 4\pi \epsilon_0 \frac{Q Q'}{r}. \quad (0.0.46)$$

Thus, since our charge  $q$  is a topological charge (and we ignore  $\epsilon_0$  or consider it separately),  $(2\pi)^2$  must be multiplied to the usual 2D Coulomb obeying  $-(1/2\pi) \log r$ .

We must also pay attention to the contribution from the ordinary spin waves, so the energy of the spin configuration can be written as

$$\mathcal{H} = J \int (\nabla \psi)^2 d^2 \mathbf{r} - 2\pi J \sum_{\langle i,j \rangle} q_i q_j \log \frac{|\mathbf{r}_i - \mathbf{r}_j|}{a} + \mu \sum q_i^2, \quad (0.0.47)$$

where  $\nabla \psi$  is the irrotational part of  $\nabla \theta$ .

### 0.0.11 Dipole dissociation and topological phase transition

At low temperatures, even if charges exist they pair up to mak dipoles, because isolated charges are costly; we need (0.0.37). Thus, no local spin rotational symmetry exists (and we have seen a quasi long range order). If the temperature is high enough free charges would be pair created.

A simple consideration is to consider the ‘free energy cost’ required to create an isolated charge. We know the energetic cost:

$$\delta E = \frac{1}{2} J \int_a^R \frac{1}{r^2} d^2 \mathbf{r} = \pi J \log \frac{R}{a}. \quad (0.0.48)$$

Here,  $a$  is the cut off (the core size of the vortex, or the lattice spacing if the model is defined on a lattice as in Fig. 31.1) and  $R$  is the system size. The entropy of the vortex is computed from the possible locations of the vortex core  $\sim (R/a)^2$ , so

$$\delta S \simeq 2k_B \log \frac{R}{a}. \quad (0.0.49)$$

Therefore, the free energy of change due to the creating of a single vortex is

$$\delta F \simeq (\pi J - 2k_B T) \log \frac{R}{a}. \quad (0.0.50)$$

If the temperature is sufficiently low, it is hard to make a vortex, but beyond

$$T_c \simeq \pi J / 2k_B \quad (0.0.51)$$

vortices could be formed easily.

Thus we have reached the picture illustrated below:

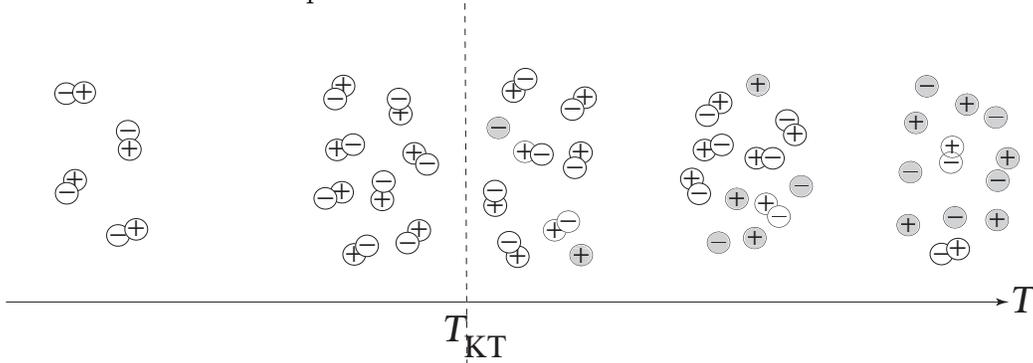


Figure 0.0.3:

However, due to the shielding effect, the Coulomb interaction energy should be different from the single charge or two bare charge interaction case as seen above.

### 0.0.12 Effective spin wave stiffness

If there are many charges, there must be the screening effect, etc. To understand the effect of vortices on the *stiffness* (= spring constant, it is microscopically  $J$ ) of spin waves, apply a constant twist so that

$$\overline{\nabla \theta} = \frac{1}{V} \int d^2 \mathbf{r} \nabla \theta = \mathbf{v}. \quad (0.0.52)$$

The effective spin wave stiffness  $J_R$  is defined as

$$\Delta F = \frac{1}{2}V J_R v^2, \quad (0.0.53)$$

where  $\Delta F$  is the free energy increase due to the twist.

$$F(\mathbf{r}) = -k_B T \log \left[ \int \mathcal{D}[\theta] \exp \left( -\beta \frac{1}{2} J \int d^2 \mathbf{r} (\nabla \theta + \mathbf{v})^2 \right) \right], \quad (0.0.54)$$

where  $\theta$  is the field constrained to be identical at the boundaries. Therefore,

$$\Delta F = \frac{1}{2}V J v^2 - \frac{1}{2}\beta J^2 \int d^2 \mathbf{r} d^2 \mathbf{r}' \langle \partial_i \theta \partial_j \theta \rangle v_i v_j + O[v^4]. \quad (0.0.55)$$

Before proceeding further, let us rewrite (0.0.55) with the aid of the Fourier component of the fields. To this end let us compute

$$\int d^2 \mathbf{r} d^2 \mathbf{r}' \langle \partial_i \theta \partial_j \theta \rangle = - \int d^2 \mathbf{r} d^2 \mathbf{s} \frac{1}{(4\pi^2)^2} \int d^2 \mathbf{q} d^2 \mathbf{q}' \langle q_i q'_j \theta(\mathbf{q}) \theta(\mathbf{q}') \rangle e^{-i(\mathbf{q} \cdot \mathbf{r} + \mathbf{q}' \cdot (\mathbf{r} + \mathbf{s}))}, \quad (0.0.56)$$

$$= - \int d^2 \mathbf{s} \frac{1}{(4\pi^2)^2} \int d^2 \mathbf{q} d^2 \mathbf{q}' \langle q_i q'_j \theta(\mathbf{q}) \theta(\mathbf{q}') \rangle 4\pi^2 \delta(\mathbf{q} + \mathbf{q}') e^{-i\mathbf{q}' \cdot \mathbf{s}}, \quad (0.0.57)$$

$$= \int d^2 \mathbf{s} \frac{1}{4\pi^2} \int d^2 \mathbf{q} \langle q_i q_j \theta(\mathbf{q}) \theta(-\mathbf{q}) \rangle e^{-i\mathbf{q} \cdot \mathbf{s}}, \quad (0.0.58)$$

$$= \int d^2 \mathbf{q} \langle q_i q_j \theta(\mathbf{q}) \theta(-\mathbf{q}) \rangle \delta(\mathbf{q}), \quad (0.0.59)$$

$$= \lim_{\mathbf{q} \rightarrow 0} q_i q_j \langle \theta(\mathbf{q}) \theta(-\mathbf{q}) \rangle. \quad (0.0.60)$$

Thus, we have

$$\frac{1}{2}V J_R v^2 = \frac{1}{2}V J v^2 - \frac{1}{2}\beta J^2 \lim_{\mathbf{q} \rightarrow 0} q_i q_j \langle \theta(\mathbf{q}) \theta(-\mathbf{q}) \rangle v_i v_j + O[v^4]. \quad (0.0.61)$$

Do we have a not isotropic  $J$ ? In our approach we take only the solenoidal part of  $\theta$ , so  $\nabla \theta$  has only one independent component. Thus,  $\langle \nabla_i \theta \nabla_j \theta \rangle = \langle \nabla \theta \cdot \nabla \theta \rangle \delta_{ij}$ . Therefore, actually, the above formula reads

$$\frac{1}{2}V J_R v^2 = \frac{1}{2}V J v^2 - \frac{1}{2}\beta J^2 \lim_{\mathbf{q} \rightarrow 0} q^2 \langle \theta(\mathbf{q}) \theta(-\mathbf{q}) \rangle v^2 + O[v^4]. \quad (0.0.62)$$

or

$$K_R = K - \frac{K^2}{V} \lim_{\mathbf{q} \rightarrow 0} q^2 \langle \theta'(\mathbf{q}) \theta'(-\mathbf{q}) \rangle. \quad (0.0.63)$$

### 0.0.13 Stiffness-charge correlation relation

Since

$$\Delta\theta(\mathbf{r}) = -2\pi\rho(\mathbf{r}), \quad (0.0.64)$$

where  $\rho$  is the charge density, (0.0.63) reads

$$K_R = K - (2\pi)^2 \frac{K^2}{V} \lim_{\mathbf{q} \rightarrow 0} \frac{\langle \rho(\mathbf{q})\rho(-\mathbf{q}) \rangle}{q^2}. \quad (0.0.65)$$

Notice that the system may be described microscopically in terms of the stiffness  $K = \beta J$  and  $\mu$  (the chemical potential of the vortices). Macroscopically, we observe renormalized quantities, so let us perform a real space renormalization group calculation.

From (0.0.65) since  $\delta(1/K) = -\delta K/K^2$  (to order  $1/K$ )

$$K_R^{-1} = K^{-1} + (2\pi)^2 \lim_{\mathbf{q} \rightarrow 0} \frac{\langle \rho(\mathbf{q})\rho(-\mathbf{q}) \rangle}{Vq^2}. \quad (0.0.66)$$

The total charge is assumed to be zero, so  $\lim_{\mathbf{q} \rightarrow 0} \rho(\mathbf{q}) = 0$ . Therefore,

$$\frac{1}{V} \langle \rho(\mathbf{q})\rho(-\mathbf{q}) \rangle = q^2 C + O[q^4]. \quad (0.0.67)$$

Thus, we have only to compute  $C$ .

$$\frac{1}{V} \langle \rho(\mathbf{q})\rho(-\mathbf{q}) \rangle = \frac{1}{V} \int d\mathbf{r} \int d\mathbf{r}' \langle \rho(\mathbf{r})\rho(\mathbf{r}') \rangle e^{i\mathbf{q} \cdot (\mathbf{r} - \mathbf{r}')}, \quad (0.0.68)$$

$$= -q_i q_j \frac{1}{2V} \int d\mathbf{r} \int d\mathbf{r}' \langle \rho(\mathbf{r})\rho(\mathbf{r}') \rangle (\mathbf{r} - \mathbf{r}')_i (\mathbf{r} - \mathbf{r}')_j + O[q^4] \quad (0.0.69)$$

Now, we use isotropy around the origin.  $(1/V) \int d^2\mathbf{r} r_i r_j = \langle \mathbf{r}^2 \rangle \delta_{i,j} / 2$

$$\frac{1}{V} \langle \rho(\mathbf{q})\rho(-\mathbf{q}) \rangle = -\frac{1}{4} q^2 \int d\mathbf{r} \langle \rho(\mathbf{r})\rho(0) \rangle |\mathbf{r}|^2 + O[q^4]. \quad (0.0.70)$$

That is,

$$C = -\frac{1}{4} \int d\mathbf{r} \langle \rho(\mathbf{r})\rho(0) \rangle |\mathbf{r}|^2. \quad (0.0.71)$$

Thus,

$$K_R^{-1} = K^{-1} + (2\pi)^2 C. \quad (0.0.72)$$

### 0.0.14 Computation of $C$

We need the charge density correlation. Next, we compute the charge correlation function for a sufficiently low free charge density case. If

$$y = e^{-\beta\mu} \quad (0.0.73)$$

is small, then the vortices are rare, and the charge must be the lowest possible, i.e.,  $\pm 1$ . There is one pair of  $\pm 1$  vortices (thus the first term is with a negative sign). Therefore, the relevant Boltzmann factor is given by

$$e^{-\beta H} = \exp \left[ 2\pi K \sum_{\langle i,j \rangle} q_i q_j \log \frac{|\mathbf{r}_i - \mathbf{r}_j|}{a} - \beta \mu \sum q_i^2 \right], \quad (0.0.74)$$

Thus the probability to have one pair of  $\pm 1$  charge distanced by  $r$  is (do not forget that there are two configurations  $+-$  or  $-+$ )

$$P = 2 \times \exp \left[ -2\pi K \sum_{\langle i,j \rangle} \log \frac{r}{a} - 2\beta \mu \right] = 2y^2 \left| \frac{\mathbf{r}}{a} \right|^{-2\pi K}, \quad (0.0.75)$$

Hence, (this is a pair of different signed charges)

$$\langle \rho(\mathbf{r}) \rho(0) \rangle = -2y^2 \left| \frac{\mathbf{r}}{a} \right|^{-2\pi K}. \quad (0.0.76)$$

Thus,  $C$  reads

$$C = \frac{1}{4} 2y^2 \int d^2 \mathbf{r} \left| \frac{\mathbf{r}}{a} \right|^{-2\pi K} \mathbf{r}^2. \quad (0.0.77)$$

This must be dimensionless, because If we honestly perform discrete calculation on a lattice, there must be a cutoff  $a$ , so finally we obtain ( $4\pi^3 = (2\pi)^2 2\pi/4$ )

$$\frac{1}{K_R} = \frac{1}{K} + 4\pi^3 y^2 \int_a^\infty \frac{dr}{a} \left( \frac{r}{a} \right)^{3-2\pi K}. \quad (0.0.78)$$

Here, dimensionally  $V$  is  $a^2$  times some number and  $J$  here is  $a^2$  times the ‘true  $a^2$ ’.

### 0.0.15 Renormalization group equation

$a$  is the cutoff or the smallest scale to describe the system. Therefore, increasing  $a \rightarrow a\ell$  (we are starting the Kadanoff construction!), we rewrite (0.0.78)

$$\frac{1}{K_R} = \frac{1}{K'} + 4\pi^3 y^2 \int_{a\ell}^\infty \frac{dr}{a} \left( \frac{r}{a} \right)^{3-2\pi K} \quad (0.0.79)$$

with

$$\frac{1}{K'} = \frac{1}{K} + 4\pi^3 y^2 \int_a^{a\ell} \frac{dr}{a} \left( \frac{r}{a} \right)^{3-2\pi K}. \quad (0.0.80)$$

This is the coarse-graining step. Rescaling (shrinking)  $a \rightarrow a/\ell$  gives ( $\mathbf{r} \rightarrow \mathbf{r}/\ell$ )

$$\frac{1}{K_R} = \frac{1}{K'} + 4\pi^3 y^2 \int_{a\ell}^\infty \frac{dr}{a} \left( \frac{r}{a} \right)^{3-2\pi K} = \frac{1}{K'} + 4\pi^3 y^2 \int_{a\ell}^\infty \frac{d(x\ell)}{a} \left( \frac{x\ell}{a} \right)^{3-2\pi K} \quad (0.0.81)$$

$$= \frac{1}{K'} + 4\pi^3 y^2 \ell^{4-2\pi K} \int_a^\infty \frac{d(x)}{a} \left( \frac{x}{a} \right)^{3-2\pi K} = \frac{1}{K'} + 4\pi^3 y'^2 \int_a^\infty \frac{dr}{a} \left( \frac{r}{a} \right)^{3-2\pi K} \quad (0.0.82)$$

Therefore,

$$y' = \ell^{2-\pi K} y. \quad (0.0.83)$$

Thus, we can find the change of the parameters according to our observation scale. Setting  $\ell = 1 + \delta\ell$ , we can obtain the differential version of the renormalization group equation.

$$\frac{dK^{-1}}{d\ell} = \frac{d}{d\delta\ell} 4\pi^3 y^2 \int_a^{a+a\delta\ell} \frac{dr}{a} \left(\frac{r}{a}\right)^{3-2\pi K} = 4\pi^3 y^2 \left(\frac{a}{a}\right)^{3-2\pi K} = 4\pi^3 y^2, \quad (0.0.84)$$

and

$$\frac{dy}{d\ell} = \frac{d}{d\delta\ell} (1 + \delta\ell)^{2-\pi K} y = [2 - \pi K] y. \quad (0.0.85)$$

Therefore,

$$\frac{dK^{-1}}{d\ell} = 4\pi^3 y^2 + O[y^4], \quad (0.0.86)$$

$$\frac{dy}{d\ell} = [2 - \pi K] y + O[y^3]. \quad (0.0.87)$$

The macroscopic stiffness is obtained by

$$K_R = \lim_{\ell \rightarrow \infty} K(\ell). \quad (0.0.88)$$

That is, the fixed point value is the value we want. Therefore, we wish to study the  $\omega$ -limit set of the RG equation.  $y = 0$ ,  $K^{-1} = \pi/2$  is a fixed point. The dynamics near the fixed point in terms of  $y$  and  $r = K^{-1} - \pi/2$  is

$$\frac{dr}{d\ell} = 4\pi^3 y^2, \quad (0.0.89)$$

$$\frac{dy}{d\ell} = \frac{4}{\pi} r y. \quad (0.0.90)$$

### 0.0.16 Renormalization flow

$$\frac{dK^{-1}}{d\ell} = 4\pi^3 y^2 + O[y^4], \quad (0.0.91)$$

$$\frac{dy}{d\ell} = [2 - \pi K] y + O[y^3]. \quad (0.0.92)$$

The macroscopic stiffness is obtained by

$$K_R = \lim_{\ell \rightarrow \infty} K(\ell). \quad (0.0.93)$$

Fixed point:  $y = 0$ ,  $K^{-1} = \pi/2$ . Let us linearize the RG equation as  $y$  and  $r = K^{-1} - \pi/2$

$$\frac{dr}{d\ell} = 4\pi^3 y^2, \quad (0.0.94)$$

$$\frac{dy}{d\ell} = \frac{4}{\pi} r y. \quad (0.0.95)$$

The flow may be shown as in Fig. 31.2.

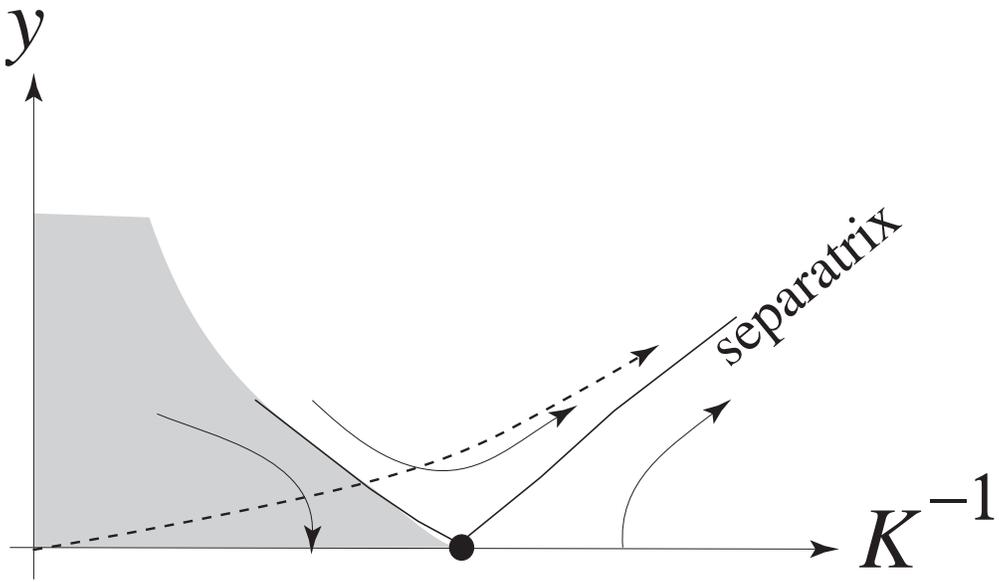


Figure 0.0.4: