26.14 Multivariate gaussian distribution

A multivariate distribution is called the *Gaussian* distribution, if any marginal distribution is Gaussian. Or more practically, we could say that if the negative log of the density distribution function is a positive definite quadratic form (apart from a constant term due to normalization) of the deviations from the expectation values, it is Gaussian:

$$f(\boldsymbol{x}) = \frac{1}{\sqrt{det(2\pi V)}} \exp\left(-\frac{1}{2}\boldsymbol{x}^T V^{-1}\boldsymbol{x}\right), \tag{0.0.1}$$

where V is the *covariance matrix* defined by (do not forget that our vectors are column vectors)

$$V = \langle \boldsymbol{x}\boldsymbol{x}^T \rangle. \tag{0.0.2}$$

If the mean value of $\boldsymbol{x}$ is nonzero $\langle \boldsymbol{x} \rangle = \boldsymbol{m}$, simply replace $\boldsymbol{x}$ in the above with $\boldsymbol{x} - \boldsymbol{m}$. The reader must not have any difficulty in demonstrating that (0.0.1) is correctly normalized.

Compute the normalization constant for the following Gaussian distribution:

$$P(x) \propto \exp\left(-\frac{1}{2}x^T V^{-1} x\right), \tag{0.0.3}$$

where x is an n-vector and V is a $n \times n$ positive definite matrix.

Since V is symmetric, we can diagonalize V with the aid of some orthogonal matrix O as $\Lambda = O^T V O$, where Λ is the diagonal matrix (i.e., $\Lambda = \lambda_1 \oplus \cdots \oplus \lambda_n$) consisting of eigenvalues of V . Introduce a new variable set (vector $\boldsymbol{y}$) through $\boldsymbol{x} = O\boldsymbol{y}$. Then,

$$\boldsymbol{x}V^{-1}\boldsymbol{x} = \boldsymbol{y}^T O^T V^{-1} O\boldsymbol{y} = \boldsymbol{y}^T \Lambda^{-1}\boldsymbol{y} = \sum y_i^2/\lambda_j. \tag{0.0.4}$$

The Jacobian for the variable change is unity, so this diagonalization allows us to compute the integral

$$\int d^n\boldsymbol{x} \exp\left(-\frac{1}{2}\boldsymbol{x}^T V^{-1}\boldsymbol{x}\right) = \prod_i \left(\int dy_i e^{-y_i^2/2\lambda_i}\right) \tag{0.0.5}$$

$$= \sqrt{(2\pi)^n detV} = \sqrt{det(2\pi V)}. \tag{0.0.6}$$

We have used $detV = \prod \lambda_i$. This is the end of the demo.