24.6 Differentiations and derivatives

We discussed the strong derivative, which may be better called *Fréchet derivative*, following the functional analysis usage.

A function $f : U \to \mathbb{R}$, where $U \subset V$ (V is a vector space),[1] is differentiable (*Fréchet differentiable*) at $x \in V$, if there is a linear map $Df_x : V \to \mathbb{R}$ so that

$$\|f(x+z) - f(x) - Df_x z\| = o[\|z\|]. \tag{0.0.1}$$

The usual partial derivative or the directional derivative (called *Gateaus derivative*) $\mathcal{D}f_x : V \to \mathbb{R}$ is defined as

$$\mathcal{D}f_x(z) = \lim_{t \searrow 0} \frac{f(x+tz) - f(x)}{t}. \tag{0.0.2}$$

If Fréchet derivative exists, then

$$\mathcal{D}f_x(z) = Df_x(z), \tag{0.0.3}$$

but if only Gateaux derivative exists, it need not be a linear map.

In this book derivatives are all in the Fréchet sense (strong). This requires an appropriate topology in the thermodynamic space. We introduce the ordinary Euclidean metric in the thermodynamic space.

Some people does not approve this approach, because thermodynamic space is spanned by physically unrelated quantities (say, V, M, etc.). However, they geometrically discuss thermodynamic space as if it is an ordinary space, so even they implicitly assume the ordinary Euclidean metric. Therefore, there is not reason to avoid the concept of the strong derivative.

Thus, we say E is once continuously Fréchet differentiable with respect to extensive variables.

[1] Generally, f is a function from a Banach space to another Banach space.