0.0.1 Classical-quantum correspondence via Wigner representation

We must compute the trace of an operator A that may be represented as a matrix in the position space:

$$\mathrm{Tr}A = \int dx \langle x|A|x\rangle = \int dx \int dx' \, \langle x|A|x'\rangle \delta(x-x'), \tag{0.0.1}$$

$$= \frac{1}{h^{Nd}} \int dx \int dx' \int dp \, \langle x|A|x'\rangle e^{-ip\cdot(x-x')/\hbar}, \tag{0.0.2}$$

$$= \frac{1}{h^{Nd}} \int dq \int dr \int dp \, \langle q+r/2|A|q-r/2\rangle e^{-ip\cdot r/\hbar}. \tag{0.0.3}$$

Here, new variables q and r are introduced as $q = (x+x')/2$ and $r = x - x'$. We introduce

$$A_W(q,p) = \int dr \, \langle q+r/2|A|q-r/2\rangle e^{-ip\cdot r/\hbar}. \tag{0.0.4}$$

This is called the *Wigner representation* of A. (0.0.3) reads

$$\mathrm{Tr}A = \frac{1}{h^{Nd}} \int dq \int dp \, A_W(q,p). \tag{0.0.5}$$

Therefore, if we can demonstrate that the Wigner representation of $e^{-\beta H}$ is close to $e^{-\beta H(q,p)}$ in the classical limit (that is, if we may assume $\hbar$ is small), we are done.
Discussion 1. 'Small' or 'large' is always a relative concept, so we must have something to compare. Here, we say $\hbar$ is small. With what do you compare it and conclude so? □

0.0.2 Wigner representation in the classical limit

We must look at the structure of the Wigner representation. Let us assume that the operator A is 'normally ordered,' so to speak. That is, all p appears on the right side (after) q and its position representation read

$$\langle x|A|x'\rangle = A\left(x, -i\hbar\frac{\partial}{\partial x}\right)\delta(x-x'). \tag{0.0.6}$$

Since

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r} + \frac{1}{2}\frac{\partial}{\partial q}, \tag{0.0.7}$$

(0.0.4) reads[1]

$$A_W(q,p) = \int dr \left[A\left(x, -i\hbar\frac{\partial}{\partial r}\right)\delta(r)\right] e^{-ip\cdot r/\hbar}, \tag{0.0.8}$$

$$= \int dr \delta(r) A\left(x, +i\hbar\frac{\partial}{\partial r}\right) e^{-ip\cdot r/\hbar} + O[\hbar], \tag{0.0.9}$$

$$= \int dr \delta(r) A(x,p) e^{-ip\cdot r/\hbar} + O[\hbar], \tag{0.0.10}$$

$$= A(q,p) + O[\hbar]. \tag{0.0.11}$$

[1] Thanks to the normal ordering, we may ignor $\partial/\partial q$ in A, and then integration by parts changes the sign of the differential operator in A when A is exchanged with the δ-function.

This combined with (0.0.5) gives what we wished to demonstrate.

Of course, the above 'demonstration' is only formal, since the relation cannot always be accurate enough; the coefficient of $O[\hbar]$ can be huge.

0.0.3 Comparison of classical and quantum free energies

We have discussed the relation between Z computed classically (with h^{3N} factor taken into account) and the true quantum Z. We know there must be corrections to the free energy for nonzero $\hbar$. Does it have a definite sign? If so, then $A \leq A_{classical}$ or not?

Discussion 1. Landau and Feynman give opposite answers. Who is correct? □