11.8 Recurrence theorem

Let T be a measure preserving map on a probability space (Ω, P). That is, $P(T^{-1}A) = P(A)$ for 'any' (measurable) $A \subset \Omega$. If $P(A) > 0$, then for almost all $\omega \in A$, the semitrajectory $\{T^n\omega\}_{n\in\mathbb{N}}$ returns infinitely many times to A. This is also true for the negative time direction as well.

[Proof] If there is a set $B \subset A$ such that $P(B) > 0$ and any semitrajectory starting from B never returns to A, then for any $\omega \in B$ and for any $n > 0$ $T^{-n}\omega \neq B$.Therefore, $\{T^{-n}B\}$ must be a sequence of disjoint sets. However, $P(T^{-n}B) = P(B) > 0$, so it is impossible. Hence, P-almost surely for $\omega \in A$ there is $n > 0$ such that $T^n\omega \in A$. We can repeat this discussion ad infinitum to show infinitely many returns.