Homework 9
due at 14:00 on November Z-day, 2009

1 I ask you to do the elementary calculations explained in the lecture notes by yourself.
   (1) Expand the chemical potential (Fermi energy) to order $T^2$ around the absolute zero for ideal fermions.
   (2) Expand the pressure $P$ of an ideal fermion gas to order $T^2$ around $T = 0$.

2 We realized that in 3D it is not always correct to use the following approximation for ideal bose gas:

\[
\langle N \rangle = \sum_{i=0}^{\infty} \langle \hat{n}_i \rangle = \int_{0}^{\infty} d\epsilon D_L(\epsilon),
\]

where $\hat{n}_i$ is the number operator denoting the occupancy number of the $i$-th one particle energy level. However, I am pretty sure that some of you are not highly convinced: Why is only the ground state special? Don’t we have to take care of not only the ground state but the next excited state(s) as well separately?; we might have to perform the following calculation, for example,

\[
\frac{\langle N \rangle}{V} = \frac{1}{V} \langle \hat{n}_0 \rangle + \frac{1}{V} \langle \hat{n}_1 \rangle + \frac{1}{V} \sum_{i=2}^{\infty} \langle \hat{n}_i \rangle.
\]

Therefore, let us perform a more honest calculation.

(1) According to our convention, the ground state energy is zero. Therefore, $\epsilon_0 = 0$. Suppose the system is in a cubic box of volume $V = L^3$. Find $\epsilon_1$ as a function of $V$.

(2) Compare the occupation number of the ground state and the first exited states (there are three such states). That is, calculate the ratio $\langle \langle \hat{n}_0 \rangle / (\langle \hat{n}_1 \rangle + \langle \hat{n}_2 \rangle + \langle \hat{n}_3 \rangle) = \langle \hat{n}_0 \rangle / 3 \langle \hat{n}_1 \rangle$ (due to degeneracy) with extremely small $\mu$ required by the BEC\(^1\). How large is it as a function of $V$?

(3) As you just saw only $\langle \hat{n}_0 \rangle$ is extensive. That is, the ground state is indeed special. Thus we need to collect the contributions of infinitely many excited states (in the large system size limit) to obtain an extensive quantity. Explain why the contribution of all the excited states can be accurately estimated by the integration in terms of the following fact: Suppose $f(i)$ is a monotone decreasing function of the positive integer $f(1) \geq f(2) \geq \cdots \geq 0$ and $\sum f(i)$ is convergent. Then,

\[
\int_{0}^{\infty} f_L(x) dx \leq \sum_{i=1}^{\infty} f(i) \leq \int_{0}^{\infty} f_U(x) dx,
\]

where $f_L(x)$ is a monotone decreasing function of $x$ such that $f_L(i-1) = f(i)$ for $i = 1, 2, \cdots$, and $f_U(x)$ is a monotone decreasing function such that $f_U(i) = f(i)$ for $i = 1, 2, \cdots$, and $f_U(0) = f_U(1)$ (draw graphs for easy understanding). \(\) I do not require any mathematical

\(^1\)which is not exactly zero but slightly negative, because the system is finite.
rigor, but an intuitive understanding.

3 Is there BEC in a 2D harmonic potential?