1 There is a $D$-dimensional ideal gas consisting of particles (bosons or fermions) whose dispersion relation is $\epsilon \propto |p|^\alpha$, where $\epsilon$ is the energy of a particle with momentum $p$.

(1) Find the density of state $D_\epsilon(\epsilon)$. You may write the surface area of the $D$-dimensional unit sphere (i.e., the surface area of the $D + 1$ dimensional unit ball) as $S_D$ (e.g., $S_2 = 4\pi$).

(2) Write down the formula for the pressure $P$.

(3) Write down the formula for the internal energy $E$.

(4) Find the ratio $PV/E$, where $V$ is the system volume.

2 Even for quantum systems the ensemble equivalence holds.

(1) To demonstrate the equivalence of microcanonical and canonical ensembles, we must demonstrate that the entropy is extensive (or more precisely, is bounded by a number proportional to the number of particles $N$), if the system Hamiltonian $H$ is bounded from below as $H \geq -NB$, where $B$ is a positive constant. Actually, we can show\(^1\)

$$W(E) \leq W_{\text{ideal}}(E + NB).$$

Thus, if we can show that the logarithm of the ideal gas microcanonical partition function $W_{\text{ideal}}(E)$ (number of all the states whose energy is not exceeding $E$) is bounded by an extensive quantity from above, we are done. Demonstrate that for bosonic systems there is such a bound:

$$W_{\text{ideal}}(E) \leq e^{N\alpha},$$

where $\alpha$ is a positive number. You may assume $E/N$ is an $N$ independent number.

(2) To demonstrate the equivalence of canonical and grand canonical ensembles, we need the upper bound of the canonical partition function for the ideal gas system (of volume $V$, number of particles $N$ and temperature $T$)

$$Z_{\text{ideal}} \leq \left( \frac{aNV}{N} \right)^N,$$

where $a$ is a positive constant. Show this for fermionic systems.\(^2\) [Hint: an easy question; do not think too much.]

3 In the following you may assume that the particles are non-interacting. Answer the following qualitative questions with justification of your answers. If no definite answer is possible, say so (with a reason).

Suppose there is a box of volume $V$ containing $N$ bosons.

\(^1\)We can actually demonstrate this with the aid of the minimax principle in the notes, but here let us accept this inequality.

\(^2\)It is false for bosonic systems. The demonstration of the canonical-grand canonical equivalence for bosonic systems is not easy.
(1) If the volume is increased under constant energy, does the temperature go down?
(2) If the volume is increased under constant entropy, does the temperature go down?
(3) Can you reduce the volume under the constant energy condition?

Next, let us consider a box of volume $V$ containing $N$ fermions. Assume that the temperature is sufficiently low (i.e., the ‘cliff’ is steep).
(4) If the volume is increased, does the temperature go down under constant energy condition?
(5) How about under the constant entropy condition?
(6) Can you reduce the volume under the constant energy condition?