HW 4 due at 11 am on Oct 1 (W), 2014.

If you submit your solutions written in (some version of) TeX reasonably before the deadline (clearly declare it is a draft; yoone@illinois.edu), you can get my response in time.\footnote{I also accept scanned hand-written drafts, IF they are neat and in a single PDF file of size no more than 0.5 MB.} You can submit your final version as a PDF file to me electronically (and will get the solution password in exchange).

No solution without your justification will get any credit. For Lecture 6 I ask you to see the simplified version.

It is strongly recommended that even for numerical problems compute your answer first symbolically, because your logic can be made explicit and because errors may be found easily.

1. There is a polymer consisting of \( N = 1,000 \) identical monomers (i.e., our polymer is a homopolymer) that can be modeled as a lattice polymer on a square lattice of edge length \( \ell = 7 \text{ Å} \).

   (1) What is the root-mean-square end-to-end distance of the polymer \( \langle R^2 \rangle \)? You may assume that the monomers are statistically independent, and can take any direction along the lattice axes with equal probabilities.

   (2) When an external electric field is applied in the positive \( x \)-direction, due to the permanent dipole moment of the monomer, the polymer elongates in the \( x \)-direction. It turns out that the monomer orientation probability is given by the following table:

   \[
   \begin{array}{c|cccc}
   \text{direction} & +x & -x & +y & -y \\
   \text{probability} & 0.5 & 0.1 & 0.2 & 0.2 \\
   \end{array}
   \]  

   (HW2.1)

   (i) What is the average end-to-end vector \( \langle R \rangle \)?\footnote{This problem is related to the study of the elasticity of the polymer.}

   (ii) What are the variances of the \( x \)-component and the \( y \)-component of the end-to-end vector \( R \)?

2. We wish to consider a Brownian particle suspended in an equilibrium fluid of temperature \( T \). Let us start with the original Langevin equation in the following form (but in the one
where \( v \) is the 1D-velocity, \( m \) is the mass of the Brownian particle, \( \zeta \) is the friction constant, and \( w \) is the noise force due to bombardment by molecules of the fluid.

(1) Assuming that the noise \( w(t) \) is given as a function of \( t \), find \( v \) as a function of time. You may assume that the initial velocity is \( v_0 \).

(2) If we wait for a sufficiently long time (that is, \( t \) is sufficiently large), the initial velocity should be totally forgotten, so in order to understand the long-time behavior we may assume \( v_0 = 0 \) without any loss of generality. After confirming this (or giving your argument for this), compute the ensemble average \( \langle v(t)^2 \rangle \) of \( v(t)^2 \) in terms of the correlation function of the noise \( \varphi(s - s') = \langle w(s)w(s') \rangle \) \((s \geq s')\), where \( \langle \rangle \) denotes the ensemble average. Notice that since the fluid in which the particle is suspended is in equilibrium, \( \varphi \) does not depend on the absolute time, but only on the time lapse between \( s \) and \( s' \).

(3) We may assume that the noise changes so randomly and so rapidly that \( w(s) \) and \( w(s') \) at different times are statistically independent and their averages are zero. Therefore, we may write

\[
\varphi(s - s') = \langle w(s)w(s') \rangle = A\delta(s - s'),
\]

where \( A \) is a positive constant (the square noise amplitude). After a long time (i.e., in the \( t \to \infty \) limit) \( v(t)^2 \) must be compatible with the equipartition of translational kinetic energy: \( \langle v^2 \rangle = k_B T/m \), so we cannot choose \( A \) arbitrarily. Find \( A \) in terms of \( k_B T \) and \( \zeta \).

3. One experiment replicating Perrin’s experiment in a modern setting uses polystyrene particles of diameter (i.e., \( 2a \)) 0.5 \( \mu \)m suspended in a buffer solution of viscosity \( \eta = 1.03 \times 10^{-3} \) Pa·s at \( T = 300 \) K. A horizontal two-dimensional stage was recorded by a microscope with a CCD camera, and its \( x \) and \( y \) coordinates are measured as functions of time. The mean square average displacement after \( t \) s in \( x \) is observed as \( \langle x^2 \rangle = 15.6 \times 10^{-13} \) \( t \) \( m^2 \) and in \( y \) is observed as \( \langle y^2 \rangle = 16.5 \times 10^{-13} \) \( t \) \( m^2 \). Assuming that you know the gas constant \( R = 8.31 \) J/mol·K, estimate Avogadro’s constant \( N_A \).

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\( ^3 \)This is also a fluctuation-dissipation relation. You might wonder why the answer is different from the fluctuation-dissipation relation we discussed in the lecture. Note the differences in the definition of the noise.