Homework 4

Due on xxx (negotiable)[1] 2018 to `minl2@illinois.edu`.
If you use pdf (scanned PDF is OK if the file size is not huge), you can submit your draft to me for comments (I will try to respond fairly quickly).

HW is treated as a learning device, so do not hesitate to ask me any question, and also you may discuss with each other (after your individual efforts). I expect all the participants will get 100%.

4.1 Inverse horseshoe.
Sketch the inverse map of the following horseshoe map (the white square to the pink shape on a disk) in Fig. H4.1.

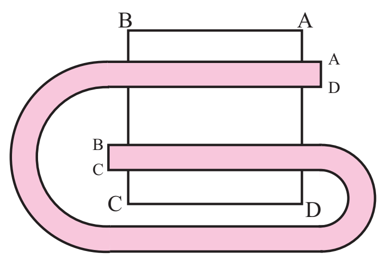

Figure H4.1: Horseshoe preserving the 'vertical order'

4.2 Skewed tent maps and their entropy.
Consider the following generally skewed tent map (Fig. H4.2):

$$x \to f_\alpha(x) = \begin{cases} x/\alpha & \text{for } x \in [0,\alpha] \\ (1-x)/(1-\alpha) & \text{for } x \in [\alpha,1] \end{cases}. \tag{H4.1}$$

This interval dynamical system $(f,[0,1])$ can have many distinct invariant measures, but only one of them is observable (i.e., reachable from a positive measure set of initial conditions). It is the flat measure $\mu = 1$ on $[0,1]$.

(1) Show that indeed this is an invariant measure.

(2) For this invariant measure compute the Kolmogorov-Sinai entropy $h_1(f_\alpha)$.

(3) (2) implies that if $\alpha \neq \beta$, then $(f_\alpha, 1, [0,1])$ and $(f_\beta, 1, [0,1])$ are not isomorphic ($\alpha, \beta \in (0,1)$). However, there is always a homeomorphism $\phi : [0,1] \to [0,1]$ that makes f_α and f_β conjugate for any $\alpha, \beta \in (0,1)$ pair. In particular, we can always find (α-dependent) continuous $\phi : [0,1] \to [0,1]$ such that ($f_{1/2}$ is the standard tent map)

$$\phi \circ f_\alpha = f_{1/2} \circ \phi. \tag{H4.2}$$

[1]You have at least two weeks

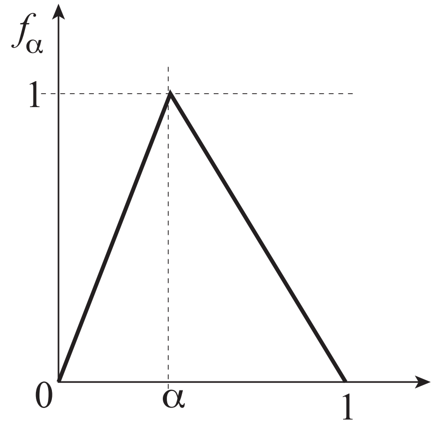

Figure H4.2: Skewed tent map

However, ϕ is, as you expect, a horrible continuous function like the devil's staircase.

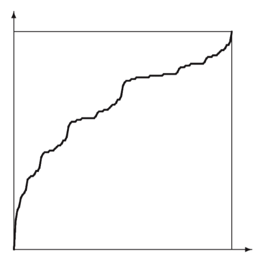

Figure H4.3: ϕ for $\alpha = 3/4$. [Fig. 2 of M Plakhotnyk, Topological conjugation of one dimensional maps, arXiv:1603.0690v1 (Mar, 2016)]

This means we can make a homeomorphism ϕ (or its inverse) to map $(f_{1/2}, 1, [0,1])$ to $(f_\alpha, \mu_\alpha, [0,1])$ with $h_{\mu_\alpha}(f_\alpha) = \log 2$. Or we can find $(f_{1/2}, \nu, [0,1])$ with the same KS entropy as $(f_\alpha, 1, [0,1])$ as computed in (2): $h_\nu(f_{1/2}) = h_1(f_\alpha)$. Knowing these facts, explain that this means that a given (topological) dynamical system $(f_\alpha, [0,1])$ has uncountably many distinct measure-theoretical dynamical systems. [Hint: it is actually trivial.]

(4) For f_α for $\alpha \in (0,1)$ its topological entropy is $\log 2$ (again trivial but explain this).

4.3 Baker's map-symbolic dynamics correspondence

We have introduce a 01 coding of the point in the unit square $Q = [0,1] \times [0,1]$ for baker's map (see Fig. 27.2). However, distinct code sequences may correspond to the same point in Q. To characterize the points having this nonuniqueness, let us make an explicit coding formula for $(x,y) \in Q$.

Looking at Fig. 27.2, we see $(x,y) \in M_{i_0} \cap T^{-1}M_{i_1} \cdots T^{-n}M_{i_n} \cdots$ indicates that x-coordinate can be coded by this 'vertical stripes'. If we code $a_{-k} = 1$ (or 0) according to $x \in T^{-k}M_1$ (or $T^{-k}M_0$), then the coding is $x \to a_0 a_{-1} \cdots a_{-n} \cdots$. Analogously, for y we can use the 'horizontal layers': If we code $a_k = 1$ (or 0) according to $y \in T^k M_1$ (or $T^{-k}M_0$), then the coding is $y \to a_1 a_2 \cdots a_n \cdots$.

(1) Compute x and y in terms of their (one-sided) sequences.

(2) Show that there are countably many points that correspond to more than one 01 sequences.