Homework 3

Due on xxx (negotiable)[1] 2018 to `minl2@illinois.edu`.
If you use pdf (scanned PDF is OK if the file size is not huge), you can submit your draft to me for comments (I will try to respond fairly quickly).

HW is treated as a learning device, so do not hesitate to ask me any question, and also you may discuss with each other (after your individual efforts). I expect all the participants will get 100%.

This Homework should not be time consuming at all.

In the following I use 'ergodicity' in an intuitive sense: all the phase states (position + moving directions are evenly covered by a typical trajectory). We will seriously discuss it when we go to the second half of the course (a more serious part).

1. Consider a 2D billiard sketched in the figure Fig. H3.1

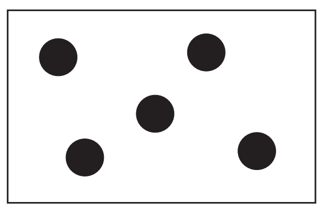

Figure H3.1: Rössler attractor

The rectangular table is $5D \times 8D$ and has 5 circular scatters of diameter D. At each collision roughly $-2\log_2 R$ bits of information about the phase point is lost. What is the information loss rate h (= the Kolmogorov-Sinai entropy = information loss/time) for this billiard table as a continuous time Hamiltonian system? You may assume[2] that the billiard is 'ergodic'.

2. In the famous 'Cloud' address[3] on the fundamental difficulty of classical physics, Lord Kelvin reconsidered the equipartition of energy to understand the specific heat ratio anomaly (that is, $\gamma = C_P/C_V > 1$ does not converge to 1 for large multiatomic gasses). Maxwell and Boltzmann assumed ergodicity (that is, all the phase points are evenly experienced by molecules) to prove the equipartition of energy, so Kelvin attacked this hypothesis as the main cause of the trouble. He even reported 'numerical simulation results' (by his assistant Mr Anderson with a ruler and sheets of paper), claiming ergodicity is violated. He discussed, for example, the following billiard tables (Fig. H3.2). From his 'experiments' he cast strong doubt about statistical mechanics. Briefly argue how Kelvin was correct (or not).

[1] You have at least two weeks
[2] Provable.
[3] "Nineteenth-Century Clouds over the Dynamical Theory of Heat and Light" April 21, 1900.

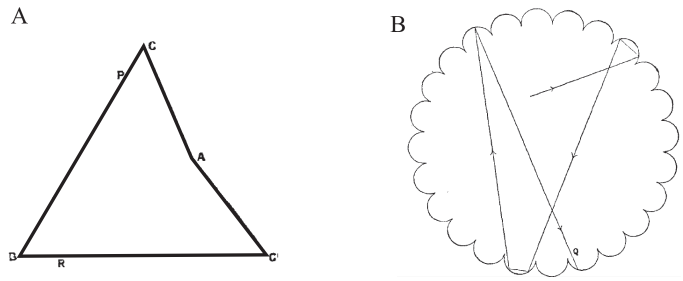

Figure H3.2: Left (A): Angles are A = 194°, B = 59°, C = C' = 53.5°. Right (B): The boundary consists of parts of circles whose centers are on a big circle. [Figs 5 and 6 of Kelvin's paper; Fig. 5 a bit cleaned]

3. An example of strange attractors simpler than Lorenz's is provided by the following set of equations: the Rössler system

$$\dot{x} = -y - z, \ \dot{y} = x + ay, \ \dot{z} = b + z(x - c). \tag{H3.1}$$

For $a = b = 0.2$ and $c = 14$ the attractor looks like Fig. H3.3.

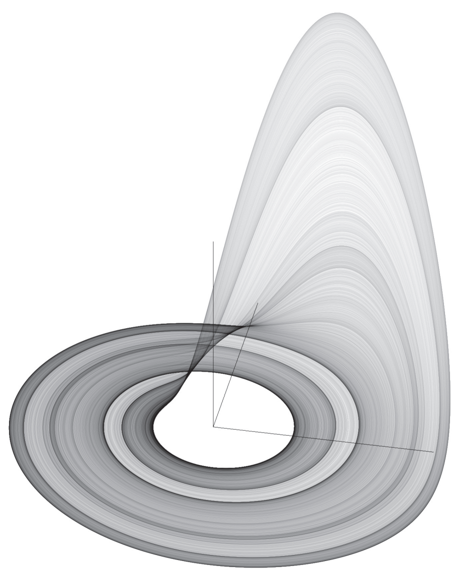

Figure H3.3: Rössler attractor [Wikipedia]

You may enjoy the following videos:

https://www.youtube.com/watch?v=o6w9CR7fk8s bifurcation[4]

https://www.youtube.com/watch?v=BAWOk9LczSU Stereoplot of a trajectory

[4]There seems to be a 'period-doubling route' to chaos, but the 'doublings' is not infinite. You can see more details from "Rössler attractor" in Wikipedia. See also C LETELLIER and V MESSAGER, "INFLUENCES ON OTTO E. RÖSSLER'S EARLIEST PAPER ON CHAOS," International Journal of Bifurcation and Chaos 20 3585 (2010).

In this problem, let us make a discrete model (based on a template) corresponding to the Rössler model, and clearly show a periodically perturbed relaxation oscillator can be chaotic.

The rule of the model is as follows:
The relaxation oscillator left alone has a strain energy accumulating speed $b > 1$, and is reset to 0 when the total energy reaches 1, but

(i) if it is hit by an external periodic signal, its speed is reduced to b^{-1};
(ii) the speed is not affected by the hits if it is already b^{-1}, but
(iii) the speed is always reset to b when the oscillator returns (when the energy reaches 1) to the lowest state.

The rule is very similar to that of Ito's earthquake model.

An example of the time evolution is in Fig. H3.4:

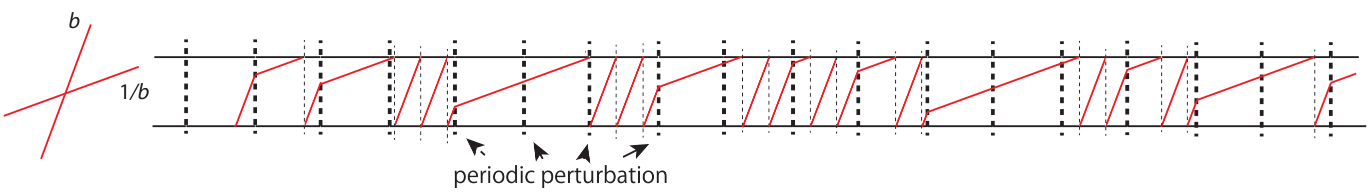

Figure H3.4: Periodically perturbed relaxation oscillator

We can plot the time evolution on the universal covering space as Fig. H3.5. As you see this is exactly the same as the Ito model.

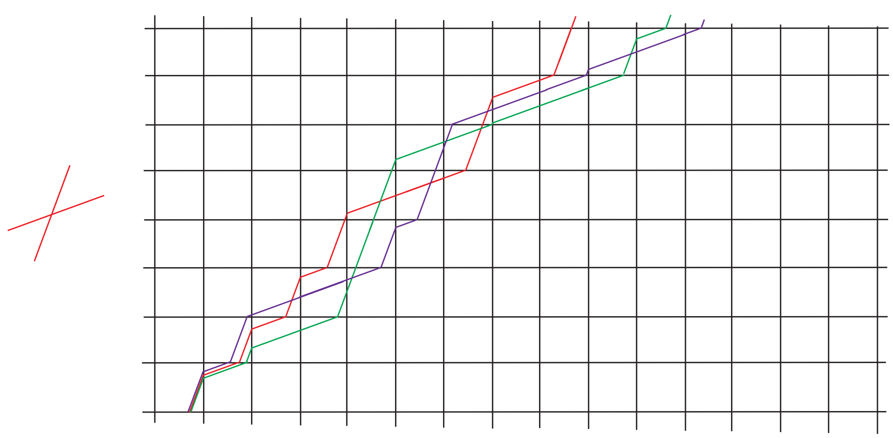

Figure H3.5: Time evolution on the universal covering space (three different initial conditions)

The rule may be visualized by using a torus (cut open; Fig. H3.6). In our case the external periodic hits need not be described, so we need one torus (square).

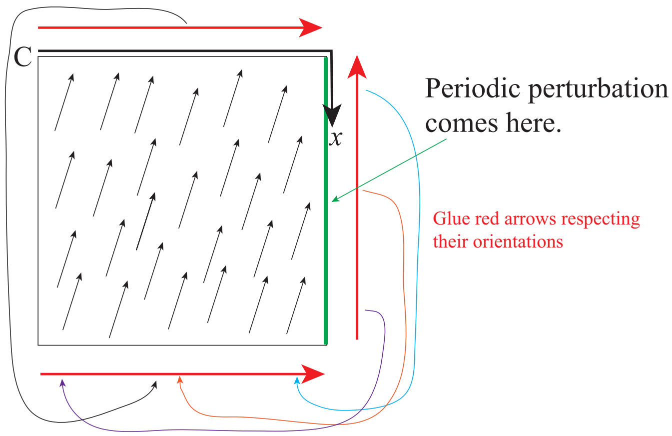

Figure H3.6: The rule; the square describes a flat 2-torus

Now we can start a topological acrobat as illustrated in Fig. H3.7.

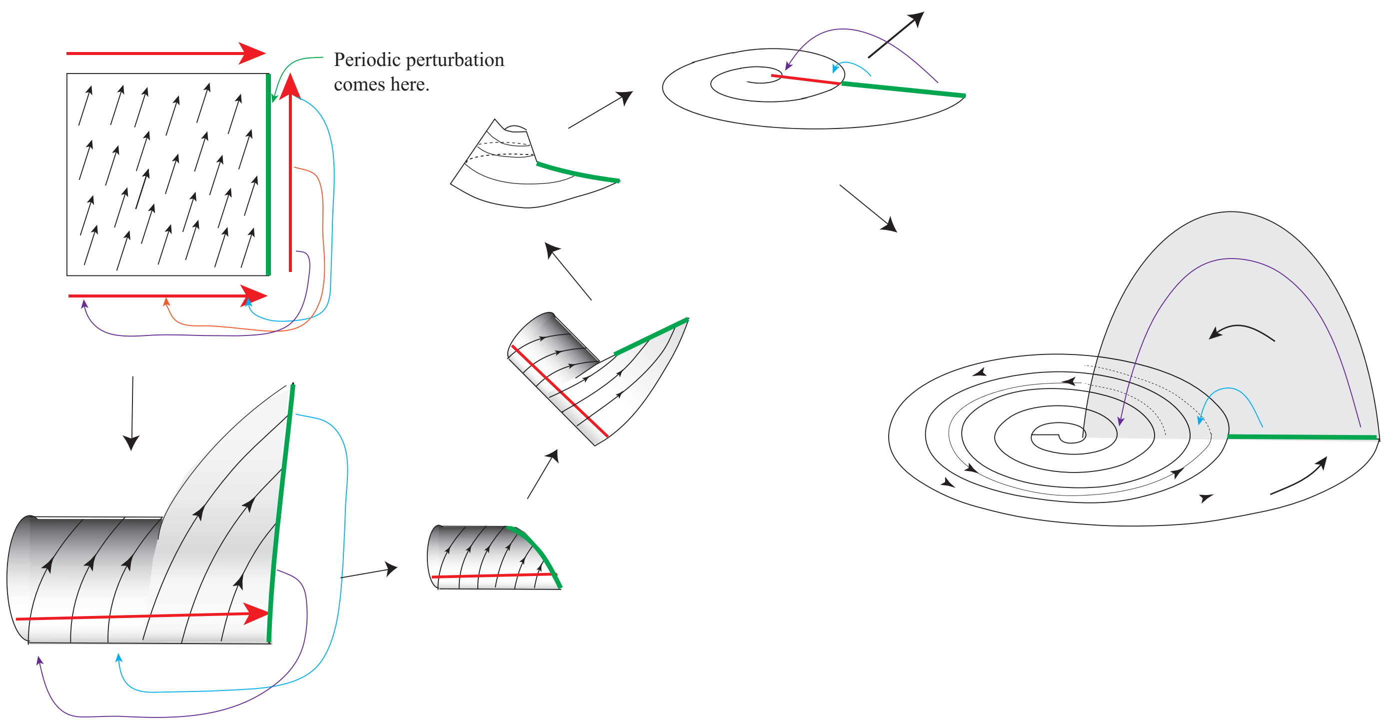

Figure H3.7: From 'billiard' to a template

3.1 To describe the dynamics of the system, we have only to record the crossing point of the upper and the right edges of the square in Fig. H3.6. We use the distance from C along these edges as the coordinate x to specify the crossing points. Make a map $F : [0, 2] \to [0, 2]$ (assume that the square is 1×1) from the n-th to the $(n + 1)$th crossings, using this coordinate system.

3.2 Prove that this map exhibits chaos. [Thus, quite unambiguously we may conclude that the original continuous template system also exhibits chaos.]