Homework 2 Solution

1 Limit cycle example

Consider the following nonlinear equation defined on $\mathbb{R}^2$ ($\epsilon > 0$):

$$\dot{x} = y + \epsilon(x - x^3), \tag{H2.1}$$

$$\dot{y} = -x. \tag{H2.2}$$

1.1. Can you show that this system exhibits a limit cycle on its phase space spanned by x and y with the aid of the Poincaré-Bendixson theorem? In this case, first argue that there is a domain D containing a disc centered at the origin such that on ∂D the vectors are all inwardly oriented (that is, no trajectory can get out from D). Then, try to apply the theorem to D.

Soln.

We must find D such that on ∂D all the vectors are inward.

Take a large circle. Note that for $|x| > 1$ the vector field is inward along the circle, because $x(1 - x^2) < 0$ (resp. > 0) for $x > 1$ (resp $x < -1$). For $|x| < 1$ consider $y > 0$ case. The slope of the vector (the red arrows in the figures below) is given by

$$\frac{\dot{y}}{\dot{x}} = \frac{-x}{y + \epsilon(x - x^3)} \tag{H2.3}$$

with $y + \epsilon(x - x^3) > y$. That is, the flow tends to go out from the $x > 0$ (the green arrow is the boundary tangent, if it is a circle), so we need a steeper declining boundary for $x > 0$ than a circle (to be transversal to the red arrow). On the other hand for $x < 0$ $y + \epsilon(x - x^3) < y$, so the vectors tend to be steeper than the tangent to the circle. Again the perturbation makes vector flow tend to go out, so we must make the boundary of D 'steeper.' Therefore, the boundary of D in the strip $|x| < 1$ must have slopes whose absolute values are larger than that of the circle(s) crossing ∂D (except at some isolated points). To illustrate such a boundary is not easy, but a sketch (exaggerated) is in Fig. H2.1:

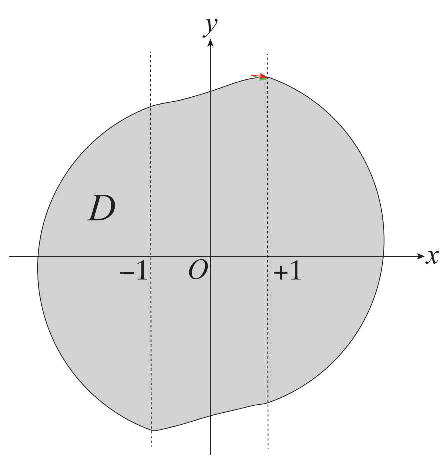

Figure H2.1: Domain D on ∂D the vector field is inward.

Let us draw the vector field (since I am lazy, I used http://user.mendelu.cz/marik/EquationExplorer/vectorfield.html). The green arrows denote the unperturbed, and the red the perturbed field. The difference is extremely exaggerated in Fig. H2.2:

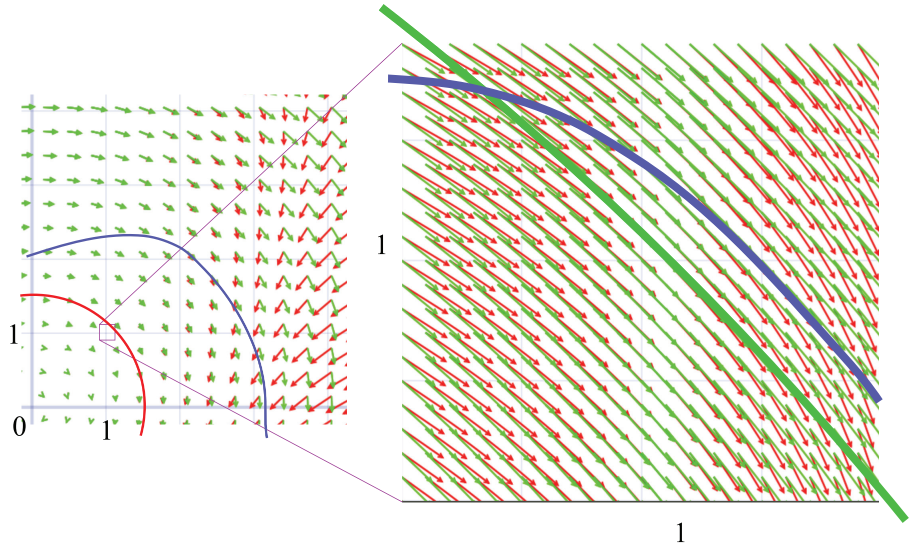

Figure H2.2: Let us look at the vector field. Green arrows denote the field with $\epsilon = 0$, Red arrows with $\epsilon > 0$ (very exaggerated). The green curve is a circle; The dark blue curve is the curve transversal to the inward red arrows, a candidate of ∂D. Clearly, any circle fails.

At least the relative shape of ∂D can be seen.

In D there is only one singularity $(0,0)$ (center for $\epsilon = 0$; for $\epsilon > 0$ it is an unstable focus (see **4.10**). Therefore, Poincaré-Bendixson's theorem tells us that the ω-limit set in D must consist of periodic orbits. HOWEVER, we cannot say the orbits are isolated, so we CANNOT assert the existence of a limit cycle (from the argument above alone) from the above topological consideration + the PB theorem alone.

1.2. Let us derive the amplitude equation using the renormalization approach and show that the system exhibits a limit cycle (relying on Chiba's theory).

Soln.

In the following I did all the calculation as 2D system using matrices, but practically, for 2 variables, the ordinary second-order 1D equation approach is often simpler. In any case, here I stick to the general approach.

We formally expand x and y in ϵ as, e.g., $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$. We have

$$\dot{x}_0 = y_0, \tag{H2.4}$$

$$\dot{y}_0 = -x_0, \tag{H2.5}$$

$$\dot{x}_1 = y_1 + \epsilon(x_0 - x_0^3), \tag{H2.6}$$

$$\dot{y}_1 = -x_1. \tag{H2.7}$$

The zeroth order is

$$x_0 = Ae^{it} + \text{ cc } . \tag{H2.8}$$

The perturbative term may be computed as

$$f(t) = x_0 - x_0^3 = Ae^{it} - 3|A|^2Ae^{it} - A^3e^{3it} + \text{ cc } . \tag{H2.9}$$

Notice that, generally,

$$\dot{z} = Az + f(t) \tag{H2.10}$$

with zero initial condition[1] reads

$$z(t) = \int_0^t ds\, e^{A(t-s)} f(s). \tag{H2.11}$$

Since

$$\begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{H2.12}$$

we have

$$\exp\left[t\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] = \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \tag{H2.13}$$

$$= \frac{1}{2}\begin{pmatrix} e^{it} + e^{-it} & -i(e^{it} - e^{-it}) \\ i(e^{it} - e^{-it}) & e^{it} + e^{-it} \end{pmatrix}. \tag{H2.14}$$

Therefore, $x_1(t)$ reads

$$x_1(t) = \frac{1}{2}\int_0^t ds\, [e^{i(t-s} + e^{-i(t-s)}] f(s). \tag{H2.15}$$

To perform our RG to get the amplitude equation, we have only to compute the secular term, so we need

$$x_1(t) = \frac{1}{2}\int_0^t ds\, e^{i(t-s)}\left(A - 3|A|^2A\right)e^{is} + \text{ cc } + \text{ regular terms} \tag{H2.16}$$

$$= \frac{1}{2}(A - 3|A|^2A)te^{it} + \text{ cc } + \text{ regular terms } . \tag{H2.17}$$

Thus

$$x(t) = Ae^{it} + \frac{\epsilon}{2}(1 - 3|A|^2)Ate^{it} + \text{ cc } + \cdots . \tag{H2.18}$$

Renormalization $t \to t - \tau$ with $A \to A(\tau)$ gives

$$x(t) = A(\tau)e^{it} + \frac{\epsilon}{2}A(1 - 3|A|^2)(t - \tau)e^{it} + \text{ cc } + \cdots . \tag{H2.19}$$

[1]We need only this case, because the deviation in the intial condition to $O[\epsilon]$ may be absorbed into the benign modification of A.

The RG equation $dx(t)/d\tau = 0$ reads (after changing $\tau \to t$)

$$\frac{dA}{dt} = \epsilon\frac{1}{2}A(1-3|A|^2). \tag{H2.20}$$

Since there is obviously a rotation in the (x, y) plane, this equation governs the size of the isolated periodic orbit = limit cycle. This (with the help of Chiba's theorem) means that the original system has a limit cycle of radius $\sim 1/\sqrt{3}$.

As you can read in Chiba's papers he automated the calculation so that you can use Matlab, Mathematica, etc. to do all the computations.

2. How the Lax pair was born:

Taste (while minimizing pain) how the Princeton group (Gardner-Greens-Kruskal-Miura) proceeded:

Let u be a solution to the KdV equation (with $-$ in front of the nonlinear term)[2]:

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0. \tag{H2.21}$$

Consider the Schrödinger equation (with $L = -\partial_x^2 + u$)

$$L\varphi = \lambda\varphi. \tag{H2.22}$$

From this we get

$$u = \lambda + \frac{1}{\varphi}\frac{\partial^2\varphi}{\partial x^2}. \tag{H2.23}$$

2.1 Put this into (H2.21), assuming λ can depend on time and φ on time and space, and check that the following equation holds (NB: Q is not an operator):

$$\lambda_t\varphi^2 + \frac{\partial}{\partial x}[\varphi Q_x - \varphi_x Q] = 0, \tag{H2.24}$$

where

$$Q = \varphi_t - B\varphi, \; B = -4\frac{\partial^3}{\partial x^3} + 6u\frac{\partial}{\partial x} + 3u_x. \tag{H2.25}$$

To save your time I allow you to use the following Lax relation:[3]

$$L_t\varphi = [B, L]\varphi \tag{H2.26}$$

in place of the KdV (H2.21). Here, L_t in our case is u_t (that is, a multiplication of u_t: $L_t g = u_t g$ for a test function g). You have only to perform operator juggling without any explicit calculation.

Soln.

[2] In order to make the accompanying Schrödinger equation to have bound states

[3] Needless to say, originally they honestly used the KdV; is there any cleaver way to get (H2.24) without using (H2.26)?

Remark. The interpretation of L_t is simply $(L(t+dt)-L(t))/dt = \partial u/\partial t$. λ is later shown to be a constant, but in this calculation we cannot show that, so λ does not commute with $\frac{\partial}{\partial t}$. Also L (u) is time-dependent, so

$$\left[L, \frac{\partial}{\partial t}\right] g = L\frac{\partial}{\partial t}g - \frac{\partial}{\partial t}Lg = L\frac{\partial}{\partial t}g - L_t g - L\frac{\partial}{\partial t}g = -L_t g. \tag{H2.27}$$

That is, $[L, \frac{\partial}{\partial t}] = -L_t$.

Note that

$$\frac{\partial}{\partial x}[\varphi Q_x - \varphi_x Q] = \varphi Q_{xx} - \varphi_{xx} Q. \tag{H2.28}$$

We may write $\partial_{xx} = u - L$, so (In the following formula L does not act beyond "] ")

$$\varphi(u-L)(\partial_t - B)\varphi - [(u-L)\varphi](\partial_t - B)\varphi = \varphi(-L)(\partial_t - B)\varphi - [(-L)\varphi](\partial_t - B)\varphi, \tag{H2.29}$$

because u is simply a multiplication operator. Since φ satisfies (H2.22), we have

$$\varphi(-L)(\partial_t - B)\varphi - [(-L)\varphi](\partial_t - B)\varphi = \varphi(-L)(\partial_t - B)\varphi - [-\lambda\varphi](\partial_t - B)\varphi. \tag{H2.30}$$

Next we use

$$\lambda\partial_t = \partial_t\lambda - \lambda_t \tag{H2.31}$$

or $\lambda\partial_t g = \partial_t(\lambda g) - \lambda_t g$ to rewrite the above relation as

$$\begin{aligned} \varphi(-L)(\partial_t - B)\varphi + \varphi(\partial_t\lambda - \lambda_t - B\lambda)\varphi &= \varphi(-L)(\partial_t - B)\varphi - \varphi(\partial_t - B)(-L)\varphi - \varphi\lambda_t\varphi & \text{(H2.32)} \\ &= \varphi[(-L)(\partial_t - B) - (\partial_t - B)(-L)]\varphi - \varphi\lambda_t\varphi. & \text{(H2.33)} \end{aligned}$$

The rest is straightforward (See te remark above)

$$\begin{aligned} &= -\varphi([L, \partial_t] - [L, B])\varphi - \varphi\lambda_t\varphi & \text{(H2.34)} \\ &= -\varphi(-L_t - [L, B])\varphi - \varphi\lambda_t\varphi. & \text{(H2.35)} \end{aligned}$$

Thus, (H2.26) (= the KdV equation) implies (H2.24). Notice that if we do not know (H2.26), we get

$$\lambda_t\varphi^2 + \frac{\partial}{\partial x}[\varphi Q_x - \varphi_x Q] = \varphi(L_t - [B, L])\varphi. \tag{H2.36}$$

2.2. We already know from the Lax-pair argument that λ is time-independent. Here, let us forget about it. Assuming that φ is smooth and localized in space, show that λ cannot depend on time: $\lambda_t = 0$.

Soln.

Since we may assume φ is L^2, integrating (H2.24) over $(-\infty, \infty)$ and noting φ and its derivative must vanish far from the origin, we get

$$\lambda_t \int dx\, \varphi^2 = 0. \tag{H2.37}$$

Thus, λ must be time-independent.

2.3 Since $\lambda_t = 0$ (H2.24) holds if $Q = 0$, we may set

$$\varphi_t - B\varphi = 0. \tag{H2.38}$$

Thus, we have three equations

$$L\varphi = \lambda\varphi, \;\; \varphi_t = B\varphi, \;\; \lambda_t = 0. \tag{H2.39}$$

From these, Lax realized his formalism. Can you derive the Lax relation from these three?

Soln.

Differentiating the first equation, we get

$$L_t\varphi + L\varphi_t = \lambda\varphi_t. \tag{H2.40}$$

Thus, we get

$$L_t\varphi + LB\varphi = \lambda B\varphi. \tag{H2.41}$$

Since λ is a number

$$L_t\varphi + LB\varphi = \lambda B\varphi = B\lambda\varphi = BL\varphi \tag{H2.42}$$

or

$$L_t\varphi = [B, L]\varphi. \tag{H2.43}$$

Notice that this is true only if φ is an eigenfunction of L. Then, why can we conclude $L_t = [B, L]$ and use as if it is an identity? A practical answer is: Recall that we use the relation to demonstrate the invariance of λ, so actually, we are safe.

A much more respectable answer is that the totality of eigenfunctions can be a basis of an L^2 space, so actually, we may[4] conclude that the operator identity is correct on the Hilbert space on which L is defined.

3 Closed orbits demand $1/r$ potential[5]

Consider a point particle (planet) in a spherically symmetric potential $V(r)$. For a given energy E and angular momentum L, the equation of motion reads in the polar coordinates

$$\frac{dr}{dt} = \pm\sqrt{\frac{2}{m}[E - V_{\text{eff}}(r)]}, \quad \frac{d\theta}{dt} = \frac{L}{mr^2}, \tag{H2.44}$$

[4] Well, of course, we differentiate φ so we must be a bit more careful; the second derivatives are also L^2 in our case. That is, honestly, we need a Sobolev space $H^2 = W^{2,2}$.

[5] We will ignore the spherically symmetric harmonic potentials.

where

$$V_{\text{eff}}(r) = \frac{L^2}{2mr^2} + V(r). \tag{H2.45}$$

3.1 Introducing $u = 1/r$, show that the above equation is equivalent to

$$\frac{1}{2}m^* \left(\frac{du}{d\theta}\right)^2 + V_{\text{eff}}(u) = E, \tag{H2.46}$$

where

$$m^* = \frac{L^2}{m}, \quad V_{\text{eff}} = \frac{1}{2}m^* u^2 + V(1/u). \tag{H2.47}$$

Soln.
Combining two equations in (H2.44), we get

$$\frac{dr}{d\theta} = \pm\frac{mr^2}{L}\sqrt{\frac{2}{m}[E - V_{\text{eff}}(r)]}. \tag{H2.48}$$

$du = -dr/r^2$ gives

$$\frac{du}{d\theta} = \mp\frac{m}{L}\sqrt{\frac{2}{m}[E - V_{\text{eff}}(r)]}. \tag{H2.49}$$

Squaring both sides, we get (H2.46).

We consider a perturbation of a circular motion with constant radius, or equivalently, $u = u_0$, which is determined by $V'_{\text{eff}}(u_0) = 0$. The perturbed orbit may be written as

$$u(\theta) = u_0 + \rho(\theta). \tag{H2.50}$$

The equation of motion around $u = u_0$ must be harmonic for small ρ, since

$$\frac{1}{2}m^* \left(\frac{d\rho}{d\theta}\right)^2 + \frac{1}{2}V''_{\text{eff}}(u_0)\rho^2 = E' = E - V_{\text{eff}}(u_0). \tag{H2.51}$$

Choosing the origin of the angle variable to be the 'farthest from the sun,' the perturbed orbit reads

$$u(\theta) = u_0 + A\cos(\Omega\theta). \tag{H2.52}$$

3.2 Show that

$$\Omega = \sqrt{\frac{3V'(r_0) + r_0 V''(r_0)}{V'(r_0)}}. \tag{H2.53}$$

(Using $V'_{\text{eff}}(u_0) = 0$, express m^* in terms of V').
Soln.
From (H2.51) we get

$$\Omega = \sqrt{V''_{\text{eff}}(u_0)/m^*}. \tag{H2.54}$$

(H2.45) implies

$$V_{\text{eff}}(u) = \frac{1}{2}m^* u^2 + V(1/u), \tag{H2.55}$$

so

$$V'_{\text{eff}}(u) = m^* u - (1/u^2)V'(1/u) \Rightarrow V''_{\text{eff}}(u) = m^* + (2/u^3)V'(1/u) + (1/u^4)V''(1/u). \tag{H2.56}$$

$V'_{\text{eff}}(u_0) = 0$ implies $m^* = (1/u_0^3)V'(1/u_0)$, so

$$V''_{\text{eff}}(u_0) = (3/u_0^3)V'(1/u_0) + (1/u_0^4)V''(1/u_0). \tag{H2.57}$$

Introducing this and m^* in (H2.54), we get

$$\Omega^2 = \frac{(3/u_0^3)V'(1/u_0) + (1/u_0^4)V''(1/u_0)}{(1/u_0^3)V'(1/u_0)} = \frac{3V'(1/u_0) + (1/u_0)V''(1/u_0)}{V'(1/u_0)}, \tag{H2.58}$$

which is (H2.53).

3.3 At the angle θ_A, where the particle is the closest to 'the sun,' $u = u_0 - A$, so $\theta_A = \pi/\Omega$. If this angle is an irrational multiple of π, we cannot expect a closed orbit. Thus, Ω must be rational. Since Ω is a continuous function of E (and L), Ω can depend on neither of them, so it is a constant (i.e., a continuous function taking only rational values must be constant). Let $f(r) = V'(r)$. Then, $\Omega = c$ (constant) implies a differential equation for f. Solving the differential equation, show that

$$f(r) = Cr^{c^2-3}, \tag{H2.59}$$

where C is an integration constant.

Soln.

The resultant differential equation reads

$$\Omega^2 = \frac{3f(r_0) + r_0 f'(r_0)}{f(r_0)} = c^2 \Rightarrow r_0 \frac{d}{dr_0} \log f(r_0) = c^2 - 3. \tag{H2.60}$$

Solving this we immediately obtain the desired $f(r_0)$.

Thus, setting $\alpha = c^2 - 2$ ($\alpha > -2$, so now $\Omega = \sqrt{\alpha + 2}$), we can write

$$V(r) = \frac{C}{\alpha} r^\alpha. \tag{H2.61}$$

Using (H2.61), (H2.46) reads

$$\frac{1}{2}m^* \left(\frac{du}{d\theta}\right)^2 + \frac{1}{2}m^* u^2 + \frac{C}{\alpha}u^{-\alpha} = E. \tag{H2.62}$$

Let us consider only the case $\alpha < 0$.

3.4 For bounded orbits $E < 0$. Now consider the $E \to 0$ limit. Set $u^{2+\alpha} = v^2$ and convert

$$\frac{1}{2}m^* \left(\frac{du}{d\theta}\right)^2 + \frac{1}{2}m^* u^2 + \frac{C}{\alpha}u^{-\alpha} = 0 \tag{H2.63}$$

into

$$\frac{1}{2}m^* \left(\frac{2}{2+\alpha}\right)^2 \left(\frac{dv}{d\theta}\right)^2 + \frac{1}{2}m^* v^2 = -\frac{C}{\alpha} > 0. \tag{H2.64}$$

This is a harmonic oscillator Hamiltonian, so (ignoring the arbitrary phase)

$$v = A\cos\Omega\theta \tag{H2.65}$$

with $\Omega = (2+\alpha)/2$ is the r-θ relation in the $E \to 0$ limit. In our case maximum r is infinite, implying $v = 0$ (i.e., $\Omega\theta = \pi/2$). Max v corresponds to minimum r (i.e., $\theta = 0$). Thus, we see $\Omega\theta_A = \pi/2$. This must be consistent with the general Ω above. Show that $\alpha = -1$ is the only solution (this is a trivial question).

Soln.
Since $u = v^{2/(\alpha+2)}$, $du/d\theta = (2/(\alpha+2))v^{-\alpha/(\alpha+2)}dv/d\theta$, so (H2.63) reads

$$\frac{1}{2}m^* \left(\frac{2}{2+\alpha}\right)^2 v^{-2\alpha/(\alpha+2)} \left(\frac{dv}{d\theta}\right)^2 + \frac{1}{2}m^* v^{4/(2+\alpha)} + \frac{C}{\alpha}v^{-2\alpha/(2+\alpha)} = 0, \tag{H2.66}$$

that is,

$$\frac{1}{2}m^* \left(\frac{2}{2+\alpha}\right)^2 \left(\frac{dv}{d\theta}\right)^2 + \frac{1}{2}m^* v^2 + \frac{C}{\alpha} = 0. \tag{H2.67}$$

The angle between max and min distances is $\Omega\theta_A = \pi/2$ or $\theta_A = \pi/(2+\alpha)$. This is the limit $E \to 0-$ case. That is, in this limit the relation between α and θ_A is given by this formula. However, in the general case this relation is, as noted in **3.3** $\theta_A = \pi/\sqrt{2+\alpha}$. For these to be compatible

$$\frac{\pi}{2+\alpha} = \frac{\pi}{\sqrt{2+\alpha}}. \tag{H2.68}$$

Thus, $\alpha = -1$.

The original paper is: S. A. Chin, “A truly elementary proof of Bertrand’s theorem,” arXiv:1411.7057v1 [physics.class-ph] 25 Nov 2014.