Homework 2

Due on xxx (negotiable)[1] 2018 to `minl2@illinois.edu`.
If you use pdf (scanned PDF is OK if the file size is not huge), you can submit your draft to me for comments (I will try to respond fairly quickly).

HW is treated as a learning device, so do not hesitate to ask me any question, and also you may discuss with each other (after your individual efforts). I expect all the participants will get 100%.

1. Limit cycle example

Consider the following nonlinear equation defined on $\mathbb{R}^2$ ($\epsilon > 0$):

$$\dot{x} = y + \epsilon(x - x^3), \tag{H2.1}$$

$$\dot{y} = -x. \tag{H2.2}$$

1.1 Can you show that this system exhibits a limit cycle on its phase space spanned by x and y with the aid of the Poincaré-Bendixson theorem? In this case, first argue that there is a domain D containing a disc centered at the origin such that on ∂D the vectors are all inwardly oriented (that is, no trajectory can get out from D). Then, try to apply the theorem to D.

1.2 Derive the amplitude equation using the renormalization approach and show that the system exhibits a limit cycle explicitly (relying on Chiba's theory).

2. How the Lax pair was born:

Taste (while minimizing pain) how the Princeton group (Gardner-Greens-Kuskal-Miura) proceeded:

Let u be a solution to the KdV equation (with $-$ in front of the nonlinear term)[2]:

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0. \tag{H2.3}$$

Consider the Schrödinger equation (with $L = -\partial_x^2 + u$)

$$L\varphi = \lambda\varphi. \tag{H2.4}$$

From this we get

$$u = \lambda + \frac{1}{\varphi}\frac{\partial^2 \varphi}{\partial x^2}. \tag{H2.5}$$

2.1 Put this into (H2.3), assuming λ can depend on time and φ on time and space, and check that the following equation holds (NB: Q is not an operator):

$$\lambda_t \varphi^2 + \frac{\partial}{\partial x}[\varphi Q_x - \varphi_x Q] = 0, \tag{H2.6}$$

[1] You have at least two weeks

[2] In order to make the accompanying Schrödinger equation to have bound states

where

$$Q = \varphi_t - B\varphi,\ \ B = -4\frac{\partial^3}{\partial x^3} + 6u\frac{\partial}{\partial x} + 3u_x. \tag{H2.7}$$

To save your time I allow you to use the following Lax relation:[3]

$$L_t\varphi = [B, L]\varphi \tag{H2.8}$$

in place of the KdV (H2.3). Here, L_t in our case is u_t (that is, a multiplication of u_t: $L_t g = u_t g$ for a test function g). You have only to perform operator juggling without any explicit calculation.

2.2 We already know from the Lax-pair argument that λ is time-independent. Here, let us forget about it. Assuming that φ is smooth and localized in space, show that λ cannot depend on time: $\lambda_t = 0$.

2.3 Since $\lambda_t = 0$ (H2.6) holds if $Q = 0$, we may set

$$\varphi_t - B\varphi = 0. \tag{H2.9}$$

Thus, we have three equations

$$L\varphi = \lambda\varphi,\ \ \varphi_t = B\varphi,\ \ \lambda_t = 0. \tag{H2.10}$$

From these, Lax realized his formalism. Can you derive the Lax relation from these three equations?

3. Closed orbits demand $1/r$ potential[4]

Consider a point particle (planet) in a spherically symmetric potential $V(r)$. For a given energy E and angular momentum L, the equation of motion reads in the polar coordinates

$$\frac{dr}{dt} = \pm\sqrt{\frac{2}{m}[E - V_{\text{eff}}(r)]},\quad \frac{d\theta}{dt} = \frac{L}{mr^2}, \tag{H2.11}$$

where

$$V_{\text{eff}}(r) = \frac{L^2}{2mr^2} + V(r). \tag{H2.12}$$

3.1 Introducing $u = 1/r$, show that the above equation is equivalent to

$$\frac{1}{2}m^*\left(\frac{du}{d\theta}\right)^2 + V_{\text{eff}}(u) = E, \tag{H2.13}$$

where

$$m^* = \frac{L^2}{m},\ \ V_{\text{eff}} = \frac{1}{2}m^*u^2 + V(1/u). \tag{H2.14}$$

[3]Needless to say, originally they honestly used the KdV; is there any cleaver way to get (H2.6) without using (H2.8)?

[4]We will ignore the spherically symmetric harmonic potentials.

We consider a perturbation of a circular motion with constant radius, or equivalently, $u = u_0$, which is determined by $V'_{\rm eff}(u_0) = 0$. The perturbed orbit may be written as

$$u(\theta) = u_0 + \rho(\theta). \tag{H2.15}$$

The equation of motion around $u = u_0$ must be harmonic for small ρ, since

$$\frac{1}{2}m^*\left(\frac{d\rho}{d\theta}\right)^2 + \frac{1}{2}V''_{\rm eff}(u_0)\rho^2 = E' = E - V_{\rm eff}(u_0). \tag{H2.16}$$

3.2 Choosing the origin of the angle variable to be the 'farthest from the sun' the perturbed orbit reads

$$u(\theta) = u_0 + A\cos(\Omega\theta). \tag{H2.17}$$

Show that

$$\Omega = \sqrt{\frac{3V'(r_0) + r_0V''(r_0)}{V'(r_0)}}. \tag{H2.18}$$

(Using $V'_{\rm eff}(u_0) = 0$, express m^* in terms of V').

3.3 At the angle θ_A, where the particle is the closest to 'the sun,' $u = u_0 - A$, so $\theta_A = \pi/\Omega$. If this angle is an irrational multiple of π, we cannot expect a closed orbit. Thus, Ω must be rational. Since Ω is a continuous function of E (and L), Ω can depend neither of them, so it is a constant (i.e., a continuous function taking only rational values must be constant). Let $f(r) = V'(r)$. Then, $\Omega = c$ (constant) implies a differential equation for f. Solving the differential equation, show that

$$f(r) = Cr^{c^2-3}, \tag{H2.19}$$

where C is an integration constant.

3.4 For bounded orbits $E < 0$. Now consider the $E \to 0$ limit. Set $u^{2+\alpha} = v^2$ and convert

$$\frac{1}{2}m^*\left(\frac{du}{d\theta}\right)^2 + \frac{1}{2}m^*u^2 + \frac{C}{\alpha}u^{-\alpha} = 0 \tag{H2.20}$$

(this is (H2.13) with $E = 0$) into

$$\frac{1}{2}m^*\left(\frac{2}{2+\alpha}\right)^2\left(\frac{dv}{d\theta}\right)^2 + \frac{1}{2}m^*v^2 = -\frac{C}{\alpha} > 0. \tag{H2.21}$$

This is a harmonic oscillator Hamiltonian, so (ignoring the arbitrary phase)

$$v = A\cos\Omega\theta \tag{H2.22}$$

with $\Omega = (2+\alpha)/2$ is the r-θ relation in the $E \to 0$ limit. In our case maximum r is infinite, implying $v = 0$ (i.e., $\Omega\theta = \pi/2$). Max v corresponds to minimum r (i.e., $\theta = 0$). Thus, we see $\Omega\theta_A = \pi/2$. This must be consistent with the general Ω above. Show that $\alpha = -1$ is the only solution (this is a trivial question).

4. Actions for the Kepler problem
(H2.11) allows us to introduce the action variables as

$$
\begin{aligned}
I_\theta &= \frac{1}{2\pi}\int_0^{2\pi} mr^2\dot{\theta}d\theta, && \text{(H2.23)} \\
I_r &= \frac{1}{\pi}\int_{r_{\min}}^{r_{\max}} \sqrt{2m[E - V_{\text{eff}}(r)]}dr. && \text{(H2.24)}
\end{aligned}
$$

Show that E is a function of $I_r + I_\theta$, if $V \propto r^\alpha$ with $\alpha = 2$ or -1 (For the harmonic case $\alpha = 2$, $E = \omega(I_\theta + I_r)$ is well-known, so you need not do anything).

You must have realized some possible relation with **3**. If you can say something general about I_r for general $V \propto r^\alpha$, I believe you can publish it.