Homework 1 Solution

The technical terms you must understand are all *italicized*.

1

Consider a *vector field* on a *chart* described in Fig. 4.5a ≡ the following Fig. HW1.1:

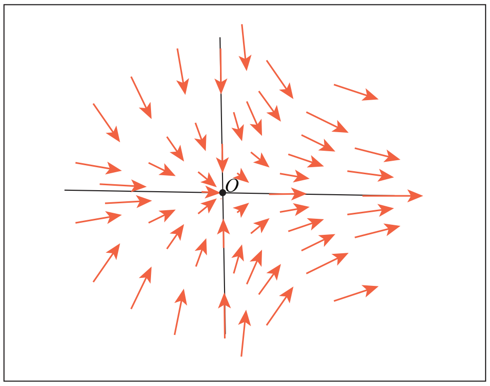


Fig. HW1.1 A vector field at a saddle-node bifurcation point

The origin is a *non-hyperbolic* (consequently, *structurally unstable*) fixed point with *index* zero as explained in Lect 4.

1-1 We wish to realize this vector field as a smooth vector field on S^2 (= 2-sphere = the surface of the ordinary 3D ball). You can use the *Poincaré-Hopf theorem* to make the simplest such vector globally defined on S^2. What other singularities does your completed global vector field have? The answer is not unique, so give two simple examples.

Solution.

The Poincaré-Hopf theorem tells us the following:

Let M be a *compact* smooth manifold. For any $X \in \mathcal{X}^r(M)$ (= the totality of C^r vector fields on M)

$$\deg(X) = \chi(M), \tag{0.0.1}$$

where $\chi(M)$ is the Euler index.

The basic facts we need are:

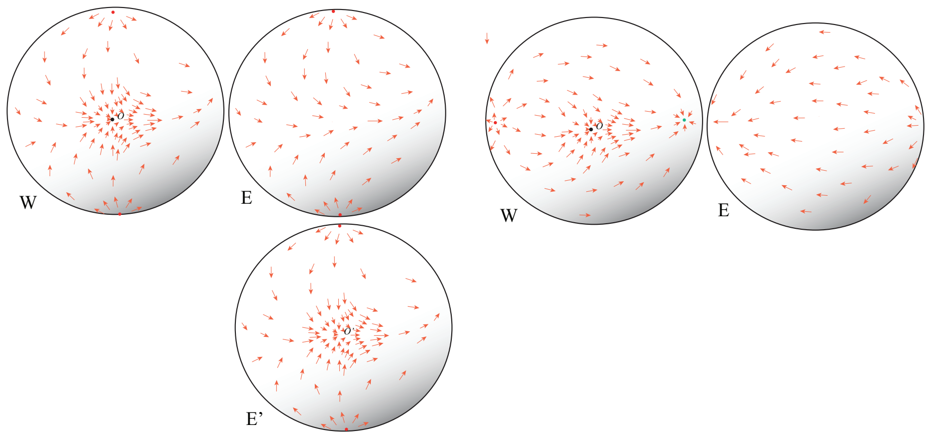
$*$ $\chi(S^2) = 2$.

$*$ deg $X = \sum_s$ index(s), where s denotes singularities of X.

$*$ The indices of (hyperbolic) *sink*s and *source*s are $+1$ and the index of a *saddle* is -1.

Although the index of the singularity in the figure is zero, $\sum_s$ index$(s) = 2$ is required. The simplest example may be with two sinks/sources. The winds blow from the south and the north hemispheres in Fig. HW1.1; we can put one source at each pole (Fig. below).

Any example will do if correct, but the next simplest example is to ‘repeat’ Fig. HW1.1: put one such singularity on the ‘western’ hemisphere and another on the eastern along the equator. A sink+source example is also illustrated below. I think there is no way to use two sinks alone.



Examples of degree 2 fields on S^2; W and E denote the western and the eastern hemispheres.

As we discussed in Lect 4, this singularity may be understood as a result of merging of a hyperbolic sink and a hyperbolic saddle, so with a small perturbation these two singularities can emerge. Or, since the index is zero, this singularity could totally disappear from the sphere as illustrated in Fig. HW1.2 Left.

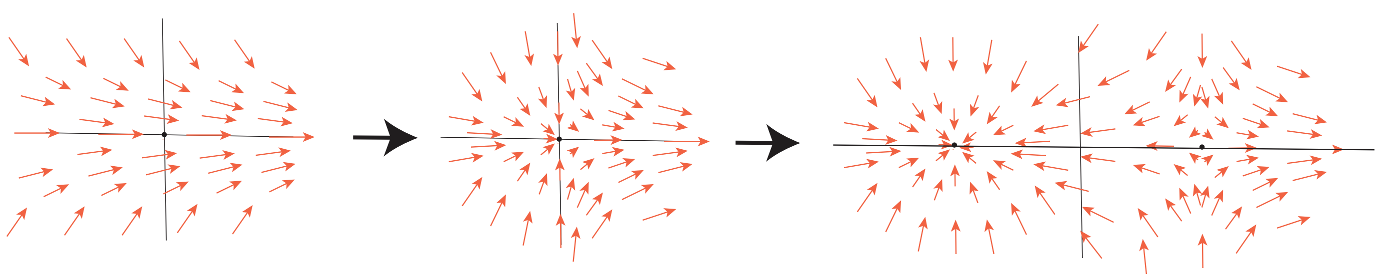


Fig. HW1.2 Saddle-node bifurcation of a 2-vector field

1-2 Suppose the linearization around the origin (the black dot in Fig. HW1.1) reads

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{0.0.2}$$

Thus, on this chart the ODE we wish to discuss has the following form:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} F(x,y) \\ G(x,y) \end{pmatrix}, \tag{0.0.3}$$

where F and G are smooth functions vanishing at the origin together with their derivatives (i.e., the Taylor expansions of F and G begin with second order polynomials. Make a *versal unfolding* of the above ODE (or the vector field) through constructing the *normal forms* of F and G in terms of polynomials with as low order as possible. The answer is virtually given in Lect 7, but I wish you to follow by yourself the '*cokernel construction*' illustrated there.[1]

Soln.

The standard strategy to make a versal unfolding of some non-hyperbolic fixed point begins with making its normal form. Truncate the result to the lowest nontrivial degree and then apply Malgrange-type argument. In practice, most known bifurcations are tabulated, but perhaps at least once in your life experiencing the simplest example may not be useless.

The most streamlined method to construct a normal form is to use Lie derivatives and Lie bracket (differential topology can tell you a very clear geometrical picture, but even without this knowledge, as is explicitly demonstrated in the notes (7.14), we know the solvability condition (= non-resonance condition) can be written as $L_A h^r = X^r$. Let us use it. The set up is:

$$L_A g = \left[-y\frac{\partial}{\partial y}, g\right] \tag{0.0.4}$$

and the general form for the higher order vector field (degree r):

$$h^r = \sum_{m+n=r} a_{mn}x^m y^n \frac{\partial}{\partial x} + \sum_r \sum_{m+n=r} b_{mn}x^m y^n \frac{\partial}{\partial y}. \tag{0.0.5}$$

You might be able to read off what you need, but here let us honestly follow the pedestrian procedure as illustrated in **7.9**.

$$[L_A, h^r] = -y\frac{\partial}{\partial y}\sum_{m+n=r}\left\{a_{mn}x^m y^n\frac{\partial}{\partial x} + b_{mn}x^m y^n\frac{\partial}{\partial y}\right\} - \sum_{m+n=r}\left\{a_{mn}x^m y^n\frac{\partial}{\partial x} + b_{mn}x^m y^n\frac{\partial}{\partial y}\right\}(-1)y\frac{\partial}{\partial y} \tag{0.0.6}$$

From this we have

$$[L_A, h^r] = \sum_{m+n=r}(-1)na_{mn}x^m y^n\frac{\partial}{\partial x} + \left\{\sum_{m+n=r}[(-1)nb_{mn}x^m y^n - b_{mn}x^m y^n]\right\}\frac{\partial}{\partial y} \tag{0.0.7}$$

The *resonance conditions* (= the conditions for the terms we cannot remove) are:

For the x component: $m+n=r$ with $n=0$. That is, $\sum_{r\ge 2} a_r x^r$.

For the y component: $m+n=r$ with $nb_{mn} = b_{mn}$ (i.e., $n=1$). $m=n=1$ for $r=2$, for $r \ge 3$ $n=1$ $m=r-1$. Thus, $Axy + B_r x^{r-1}y$, where A and B_r are real numbers.

Thus, the lowest nontrivial polynomial terms that we cannot remove are: for the x component x^2 and for the y component xy. Therefore, the normal form we use to construct a versal unfolding is

$$\dot{x} = Ax^2, \tag{0.0.8}$$

$$\dot{y} = -y + Bxy. \tag{0.0.9}$$

Malgrange's theorem tells us, adding lower order polynomials to the above normal form gives a versal unfolding:

$$\dot{x} = a + bx + Ax^2, \tag{0.0.10}$$

$$\dot{y} = c - y + Bxy. \tag{0.0.11}$$

[1] As noted during the lecture, in practice, with two variables x and y, most popular normal forms and their versal unfoldings are tabulated, so look for what you need.

We can remove bx as explained in **7.5**. c simply shifts the singularity position without qualitatively changing the field. Thus, probably the most convenient versal unfolding is (even the number of parameters could be different between different versal unfoldings)

$$\dot{x} = \nu + Ax^2, \tag{0.0.12}$$

$$\dot{y} = -y + Bxy. \tag{0.0.13}$$

In the case of Fig. HW1.1, A is positive and ν changes its sign from $+$ to $-$ (B does not matter very much locally).

A bit more careful argument is: By shifting y we can remove c. However, this produces a term proportional to x in $\dot{y}$. However, this term (needless to say, we must assume c is not huge) does not change the topology of the vector field. To remove bx we must shift x a bit. This produces an extra term proportional to y in $\dot{y}$, but this does not change the strong stabilizing term $-y$, so we eventually justify the above versal unfolding. As noted, you may use different forms with different number of parameters as another versal unfolding, but in practice, we should minimize the number of 'critical' (= topology changing) parameters. □

2

We know the vector field illustrated in Fig. HW1.1 can be globally realized on T^2. We can cut and open T^2 as illustrated in Fig. HW1.3. The field A is without any singularity. As you see it satisfies the periodic boundary conditions, so it is realizable on T^2. We perform a *surgery* in the disc bounded by a dotted circle to smoothly embed any field illustrated in Fig. HW1.2, so we can realize a *saddle-node bifurcation* in a flow on T^2.

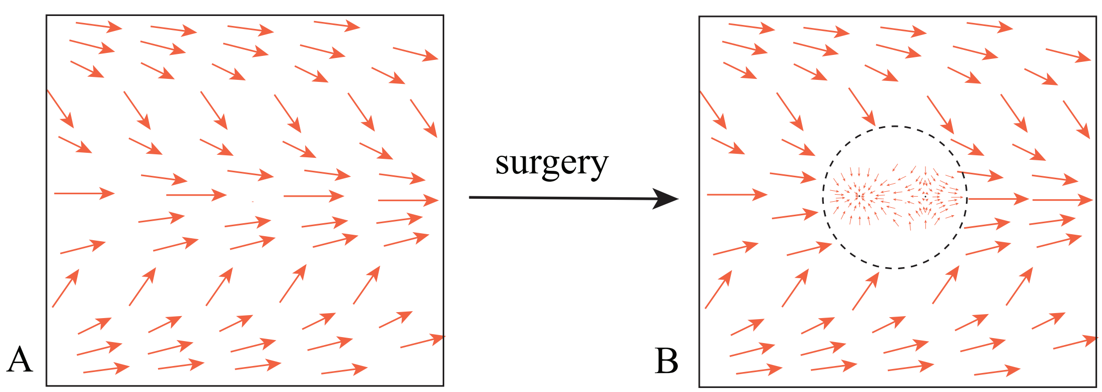


Fig. HW1.3 Realization of a saddle-node bifurcation on T^2. The Euler index of T^2 is zero, so we can do this without any other singularities.

Up to this point we have interpreted the figure as vector fields and considered the flows defined by them. However, we can also interpret the arrows as displacements due to a smooth map from T^2 into itself as illustrated in Fig. HW1.4.

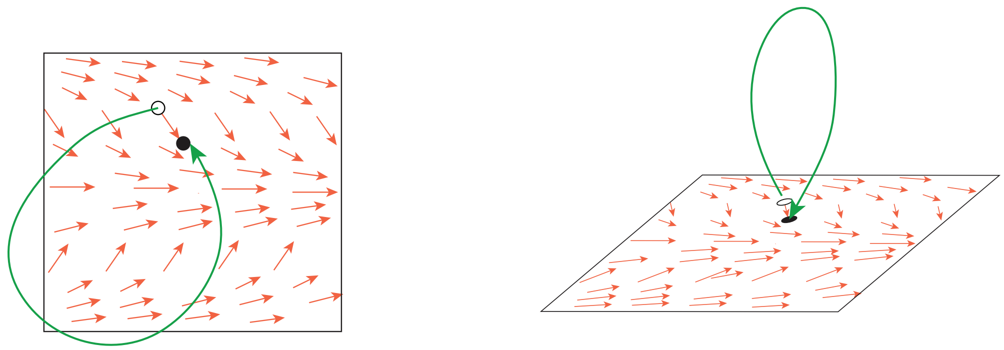


Fig. HW1.4 A *diffeomorphism* on T^2 whose displacement vectors are given by red arrows. The map (green curved arrow) maps a white dot to a black dot. The green curve can be realized as a flow from T^2 to T^2.

Thus we can make a flow on T^3 (3-torus) whose *Poincaré map* is just given by the vector field (as displacements) such as B in Fig. HW1.3.

After a saddle node bifurcation we can perform a surgery near the emergent sink to realize a *horseshoe map*, because it can be constructed on a 2-disk D as illustrated (note all the arrows exhibiting displacements by the diffeomorphism can be drawn inwardly) in Fig. HW1.5 as a diffeomorphism.

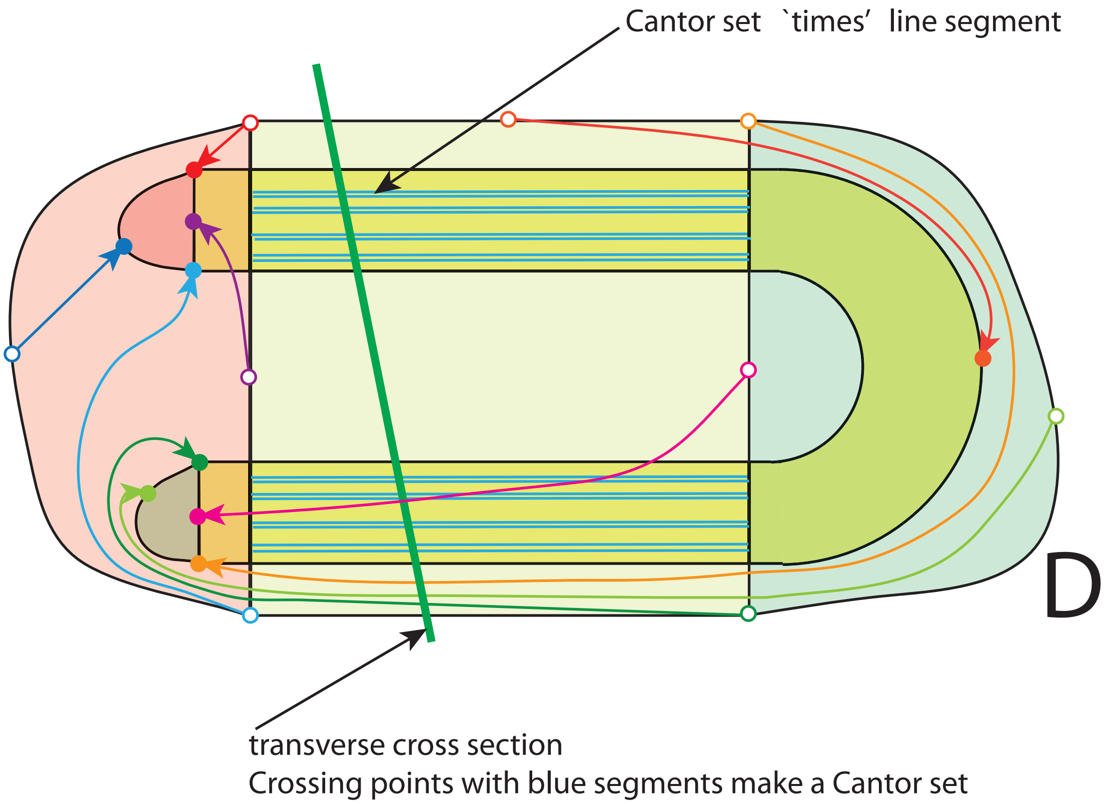


Fig. HW1.5 Horseshoe realized as a map from 2-disk D into itself. Notice that the displacements (denoted by 'rainbow-colored' curved arrows from open dots (tiny disks) to filled dots) due to the map can be drawn inside D without mutual crossing. Thus, this map is realizable by a continuous deformation of D in a plane, so we can *suspend* it into a flow so that the map illustrated here is realizable as its Poincaré map.

Therefore, we can construct a flow (by suspension) on T^3 (3-torus) that has a horseshoe in its Poincaré map ϕ. Thus, as we have already seen, the yellow square region contains a set Q = a Cantor set $\otimes$ a line segment (the blue lines in Fig. HW 1.5) whose image by ϕ always contain itself (that is, $\phi(Q) \supset Q$ (Q is not an invariant set, since numerous points are expelled from Q by the map; we could say we can make an invariant foliation).

2.1 Where do all the points in the disk D that cannot stay in Q go? [This tells you, unfortunately, that a complicated motion (which definitely exists) in our system is not observable, generally.]

Soln

As can be seen from Fig. HW1.5 The red 'semidisk' (the left end) goes into itself; the green 'semidisk' (the right end) goes into the red semidisk. The yellow square is stretched and can stay there only when the stretched portions overlap with the original square. The rest goes into red or green portion (especially the bent portion in the right green semidisk is sent into the left red semidisk). Thus, except for the points on the blue segments Q all other points are eventually sucked into the red 'semidisk.' What would happen to the point in this red region? The simplest case is that there is a sink. Of course, we could have a stable limit cycle there (but definitely no chaos).□

2.2 The transversal cross section of Q (crossing of blue line segments with the green segment in Fig. HW1.5) happens to be modeled by an 'excessively steep tent map (f; Fig. 2.8 ≡ HW1.6 below) with the slope $\pm(2+\alpha)$, where $\alpha > 0$.

What is the *Minkowski (= box) dimension* of this *Cantor set*?

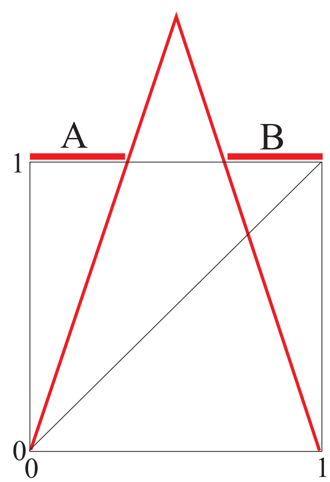


Fig. HW1.6 A 'too tall' tent map f with slope $2+\alpha$ or $-(2+\alpha)$. The red closed intervals are named A and B, respectively. Just as the yellow square in Fig. HW1.5, almost all the points ion $[0,1]$ is lost to $-\infty$.

Soln.
After n applications of the map, the remaining set consists of 2^n closed segments of length $(2+\alpha)^{-n}$. We can proceed roughly as follows. if $\delta \sim (2+\alpha)^{-n}$, then $N_n = 2^n$. Therefore, the box counting dimension is (since all the segments have the same size)

$$\dim_M = \frac{\log 2^n}{\log(2+\alpha)^n} = \frac{\log 2}{\log(2+\alpha)}. \tag{0.0.14}$$

The formula indeed gives the correct answer for 'THE' (middle third removal) Cantor set with $\alpha = 1$. Obviously, this must be a decreasing function of α. □

2.3 Within this modeling the blue lines Q (or its cross section with the green line in Fig. HW1.5) in the yellow square may be marked with a binary sequence depending on $f^n(x) \in$ A or B in Fig. HW1.6 as follows, where $x \in [0,1]$:

$x = a_0 a_1 \cdots a_n \cdots$ ($a_i \in \{0,1\}$), where (for example; different codings will do)

$a_n = 0$, if $f^n(x) \in$ A and

$a_n = 1$, if $f^n(x) \in$ B.

Let us confirm rather trivial statements:

(i) There are periodic orbits with any period.

(ii) Let $q = b_1 b_2 \ldots$ be a binary expansion expression of 'Iliad' (assume some version is specified). Is there any such line in Q?

(iii) Can you have a periodic point indefinitely close to any given non-periodic point in this invariant set?

No precise proof or demonstration is needed, but give an intuitive supporting argument for your answer.

Soln.

(i) Binary rational numbers generally correspond to periodic orbits, because

$$f(a_0 a_1 \cdots a_n \cdots) = a_1 a_2 \cdots a_n \cdots. \tag{0.0.15}$$

That is, f corresponds to a *shift* in the coded space.

(ii) Yes, any binary number sequence must appear in some irrational 01 sequence (i.e., the binary expansion of an irrational number in $[0, 1]$). Since Iliad is of finite length, we only need a certain rational number corresponding to the coded sequence infinitely repeatedly concatenated, or a trajectory eventually sucked into the origin by padding the coded sequence with infinitely many 0s. You can imagine a trajectory corresponding to the totality of poems ever produced (and that will ever be produced by the end of the universe); after all it is a finite sequence.

(iii) Notice that if x and x' expanded as specified are very close, if the binary expansions agree up to a large n. Thus, we can introduce a topology in the totality of binary sequences such that it is equivalent to the usual topology on $[0, 1]$. Thus, the answer to (iii) is trivially yes.

A bit more carefully, note that the distance between x and x' whose code sequences differ only beyond the n-th letter. Then, the distance between these points in the real world is something like a^{-n} for some $a > 1$. (I guess $a = 3$ seems OK, correct?)