Homework 1

Due on xxx (negotiable)[1] 2018 to `minl2@illinois.edu`.
If you use pdf (scanned PDF is OK if the file size is not huge), you can submit your draft to me for comments (I will try to respond fairly quickly).

HW is treated as a learning device, so do not hesitate to ask me any question, and also you may discuss with each other (after your individual efforts). I expect all the participants will get 100%.

1

Consider a vector field on a chart described in Fig. 4.5a ≡ the following Fig. HW1.1:

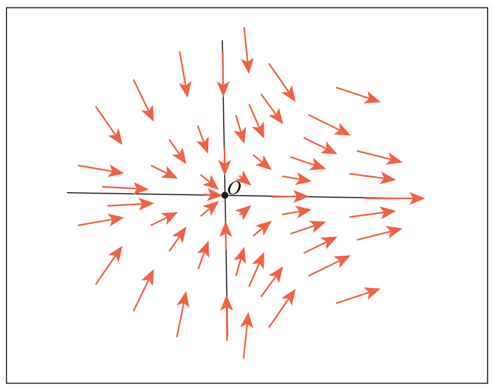

Fig. HW1.1 A vector field at a saddle-node bifurcation point

The origin is a non-hyperbolic (consequently structurally unstable) fixed point with index zero as explained in Lect 4.

1-1 We wish to realize this vector field as a global smooth vector field on S^2 (= 2-sphere = the surface of the ordinary 3D ball). You can use the Poincaré-Hopf theorem to make the simplest such vector globally defined on S^2. What other singularities does your completed global vector field have? The answer is not unique, so give two simple examples.

As we discussed in Lect 4, this singularity may be understood as a result of merging of a hyperbolic sink and a hyperbolic saddle, so with a small perturbation these two singularities can emerge. Or, since the index is zero, this singularity could totally disappear from the sphere as illustrated in Fig. HW1.2.

1-2 Suppose the linearization around the origin (the black dot in Fig. HW1.1) reads

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{0.0.1}$$

[1]You have at least two weeks

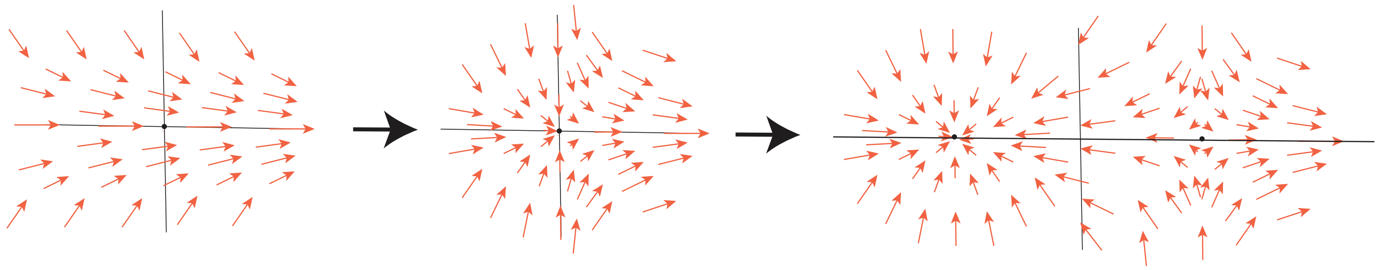

Fig. HW1.2 Saddle-node bifurcation of a 2-vector field

Thus, on this chart the ODE we wish to discuss has the following form:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} F(x,y) \\ G(x,y) \end{pmatrix}, \tag{0.0.2}$$

where F and G are smooth functions vanishing at the origin together with their derivatives (i.e., the Taylor expansions of F and G begin with second order polynomials). Make a versal unfolding of the above ODE (or the vector field) through constructing the normal forms of F and G in terms of polynomials with as low orders as possible. The answer is virtually given in Lect 7, but I wish you to follow by yourself the 'cokernel construction' illustrated there.

2

We know the vector field illustrated in Fig. HW1.1 can be globally realized on T^2. We can cut and open T^2 as illustrated in Fig. HW1.3. The field A is without any singularity. As you see it satisfies the periodic boundary conditions, so it is realizable on T^2. We perform a surgery in the disc bounded by a dotted circle to smoothly embed any field illustrated in Fig. HW1.2, so we can realize a saddle-node bifurcation in a flow on T^2.

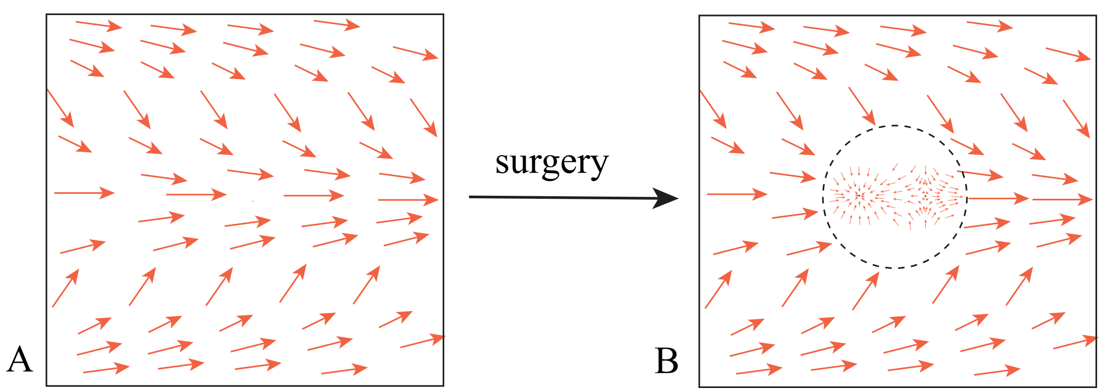

Fig. HW1.3 Realization of a saddle-node bifurcation on T^2

Up to this point we have interpreted the figure as vector fields and considered the flows defined by them. However, we can also interpret the arrows as displacements due to a smooth map from T^2 into itself as illustrated in Fig. HW1.4.

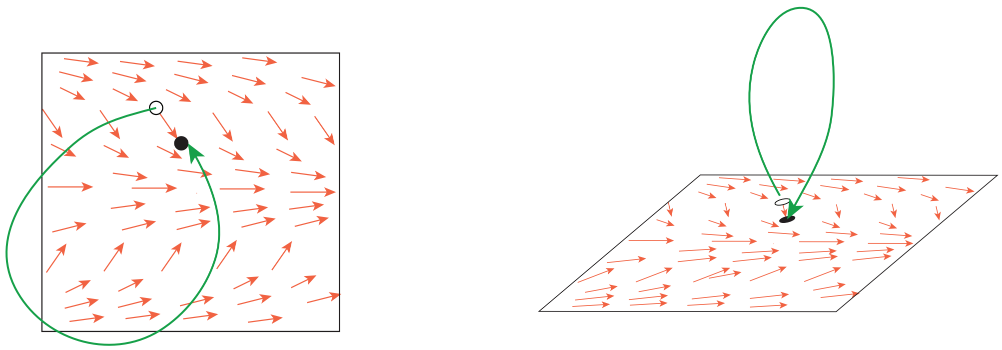

Fig. HW1.4 A diffeo on T^2 whose displacement vectors are given by red arrows. The map (green curved arrow) maps a white dot to a black dot.

Thus we can make a flow on T^3 (3-torus) whose Poincaré map is just given by the vector field (as displacements) such as B in Fig. HW1.3.

After a saddle node bifurcation we can perform a surgery near the emergent sink to realize a Horseshoe map, because it can be constructed on a 2-disk D as illustrated in Fig. HW1.5 as a diffeomorphism.

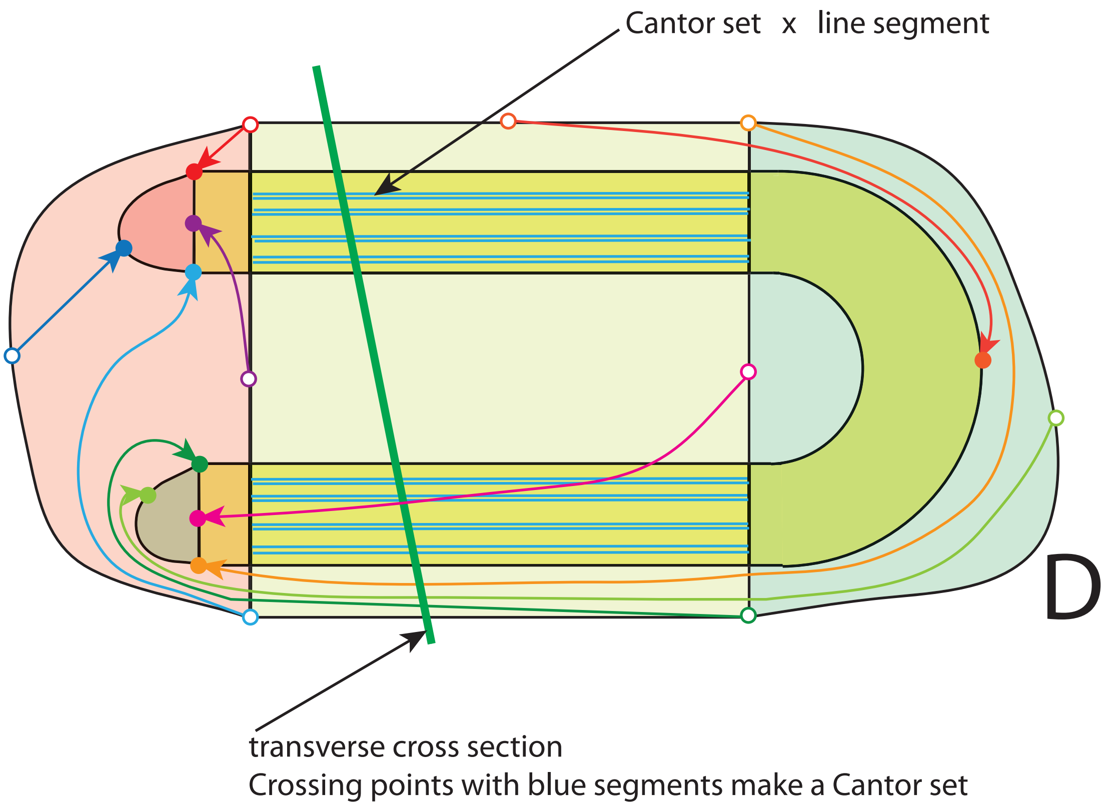

Fig. HW1.5 Horseshoe realized as a map from 2-disk D into itself. Notice that the displacements (denoted by 'rainbow-colored curved arrows from open disks to filled disks) due to the map can be drawn inside D without mutual crossing. Thus, this map is realizable by a continuous deformation of D in a plane, so we can suspend it into a flow.

Therefore, we can construct a flow (by suspension) on T^3 (3-torus) that has a horseshoe in its Poincaré map ϕ. Thus, as we have already seen, the yellow square region contains a set Q = a Cantor set $\otimes$ a line segment (a bunch of blue line segments in Fig. HW 1.5) whose image by ϕ always contain itself (that is, $\phi(Q) \supset Q$ (Q is not an invariant set, since numerous points are expelled from Q by the map).

2.1 Where do all the points in the disk D that cannot stay in Q go? [This tells you, unfortunately, that a complicated motion (which definitely exists as seen in **2.3**) in our system is not observable, generally.]

2.2 The transversal cross section of Q (crossing of blue line segments with the green segment in Fig. HW1.5) happens to be modeled by an 'excessively steep tent map $f : \mathbb{R} \to \mathbb{R}$ (Fig. 2.8 $\equiv$ HW1.6 below) with the slope $\pm(2 + \alpha)$, where $\alpha > 0$.

What is the Minkowski (= box) dimension of this Cantor set?

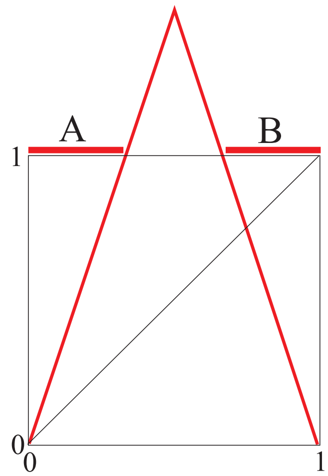

Fig. HW1.6 A 'too tall' tent map with slope $2+\alpha$ or $-(2+\alpha)$ (only the portion on $[0,1]$ is illustrated). The red closed intervals are named A and B, respectively.

2.3 Within this modeling the blue lines Q (or its cross section with the green line in Fig. HW1.5) in the yellow square may be marked with a binary sequence depending on $f^n(x) \in$ A or B in Fig. HW1.6 as follows, where $x \in [0,1]$:

$x = a_0 a_1 \cdots a_n \cdots$ $(a_i \in \{0,1\})$, where

$a_n = 0$ if $f^n(x) \in$ A and

$a_n = 1$ if $f^n(x) \in$ B.

Let us confirm rather trivial statements:

(i) There are periodic orbits with any period.

(ii) Let $q = b_1 b_2 \ldots$ be a binary expansion expression of 'Iliad' (assume some version is specified). Is there any such line segment in Q?

(iii) Can you have a periodic point indefinitely close to any given non-periodic point in this invariant set?

No precise proof or demonstration is required, but give an intuitive supporting argument for your answer.