P510 Final

Due Midnight Dec 15, 2018 (submit electronically to YO).

Part I.

You are asked to write succinct (no more than 10 lines with 'reasonable' fonts) explanations of the following technical terms with (if possible) definitions, significance, examples, warnings, etc.

(1) Structural stability

A property of a mathematical entity in a topological space (usually in the normed space) is structurally stable if all the members of all the sufficiently small neighborhoods of the entity share the property. It is a mathematical expression of stability against meaningful perturbations of the dynamics. Originally, the concept was introduce by Andronov and Pontryagin, and then very seriously pursued by Peixoto in dynamical systems: for a smooth vector fields or maps, Axiom A (hyperbolicity of the nonwandering set without saddle connection) & strong transversality is a necessary and sufficient condition for structural stability. Unfortunately, structurally stable systems do not make a dense set (although trivially open) in higher (≥ 3 for flows) dimensional spaces.

(2) Ergodicity (you may assume 'invariant measure' already explained)

Let (f, μ, M) be a dynamical system with an invariant measure μ for a map f that maps a (smooth) manifold M into itself (here f may be a time evolution operator ϕ_t for a flow at time t). For any μ-measurable set A, if $A = f^{-1}A$ (modulo μ-measure zero sets), we say A is invariant. If all the invariant sets A satisfy $\mu(A)(1-\mu(A)) = 0$, we say μ is ergodic. The most important properties of an ergodic dynamical system are the agreement of the time average and the average over μ (Birkhoff's theorem) and the recurrence: μ-almost every state of a dynamical system returns to arbitrarily close to itself infinitely many times (no irreversibility is possible for a closed system as Zermelo pointed out). However, the needed time for recurrence is, as Boltzmann emphasized, hyper-astronomically big.

(3) ϵ-tracing (property).

Let $q = \{y_n\}$ be an approximate trajectory. Suppose there is an exact trajectory $\{x_n = f^n(x_0)\}$ satisfying $\|x_n - y_n\| < \epsilon$ for all n (in a certain time span T), where f is a map defining the dynamics of the system, we say the approximate trajectory q is ϵ-traced by $\{x_n\}$. We say q has an ϵ-tracing property (or ϵ-traceable, shadowable). If a system is sufficiently chaotic (say, Axiom A), any α-pseudo orbit $q = \{y_n\}$ satisfying $\|y_n - f(y_{n-1})\| < \alpha$ for sufficiently small $\alpha > 0$ for all $n \in T$ is ϵ-traceable for a given ϵ. If a system is shadowable, it has a Markov partition (and vice versa). That is, only if a system is sufficiently chaotic (i.e., isomorphic to a Bernoulli-system) can a numerical solution capture exact trajectories, although we cannot tell which initial conditions give the right trajectories. It is very ironic that if a trajectory is stable, no tracing property is guaranteed.

Part II.

Write succinct but 'intelligent-lay-person-friendly' short articles (no more than 10 lines with 'reasonable' fonts) with the following titles:

(1) What is chaos?

Chaos is a collective property of a deterministically time-evolving system for which most trajectories are in the long run random (unpredictable). In chaos doubling the accuracy to specify the initial condition only extends the time span of reliable prediction by a finite additive amount (say, 1 sec more; thus to extent for 10 sec, we need accuracy of 2^{-10}). Chaos was sensationally received by the 'new age complexity buffs' (including the so-called complexity scientists) as something fundamentally important to the complexity of the world (UIUC was not an exception, unfortunately), leading to total confusion of complexity and mere complication; even a Nobel laureate was fooled (or a stupid laureate was exposed). However, the theory is firmly based on the unique existence of the future as a function of the initial condition; it is a part of classical physics; thus chaos was clearly recognized by Poincaré more than a 100 years ago.

(2) What is a measure-theoretical dynamical system?

A dynamical system (f, M) is a rule f to determine the future state (position) of an entity (point) on the stage (in a space) M uniquely in terms of its current state. We can study the collection (ensemble) of many points obeying this dynamical rule simultaneously. We could imagine a cloud consisting of such points wandering on the stage. To study how this cloud changes is the purpose of measure-theoretical dynamical systems. Here, 'measure' may be understood as a probability that tells us how likely we find points in a certain locality of the stage. Some clouds are time independent, which are called invariant measures. (f, M) with a compatible invariant measure is called a measure-theoretical dynamical system. It corresponds to a particular steady state of the undelying dynamical system.

Part III.

Make a Markov partition of T^2 (illustrated on its universal covering space) for the following toral diffeomorphism $T_A : T^2 \to T^2$ with

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \tag{H4.1}$$

Hint: Mimic **40.6**.

Soln.

We must determine the stable and unstable manifold of the fixed point (the origin). We need eigenvalues and corresponding eigenvectors for A. The eigenvalues are

$$\lambda_\pm = \frac{1 \pm \sqrt{5}}{2} \simeq -0.618, 1.618. \tag{H4.2}$$

The eigendirections are specified by, e.g., vectors $\left((\sqrt{5} \mp 1)/2, 1\right) \simeq (0.618, 1)$ or $(1.618, 1)$ (respectively). Notice that $\lambda_+\lambda_- = -1$.

Therefore, the stable (+) and the unstable (−) manifolds of the origin are given by

$$y = \lambda_\pm^{-1} x \simeq -1.618x, \text{ or } 0.618x. \tag{H4.3}$$

The position a in the figure below is given by (do not forget that $\lambda_- < 0$)

$$\left(\frac{\lambda_+\lambda_-}{\lambda_- - \lambda_+}, \frac{\lambda_-}{\lambda_- - \lambda_+} \right), \tag{H4.4}$$

and a'

$$\left(\frac{-\lambda_+}{\lambda_- - \lambda_+}, \frac{-1}{\lambda_- - \lambda_+} \right), \tag{H4.5}$$

Notice that a $\times$ λ_+ is a', because $\lambda_+\lambda_- = -1$. I will not do all such calculations for other vertices, but you see the expansion rate is correct.

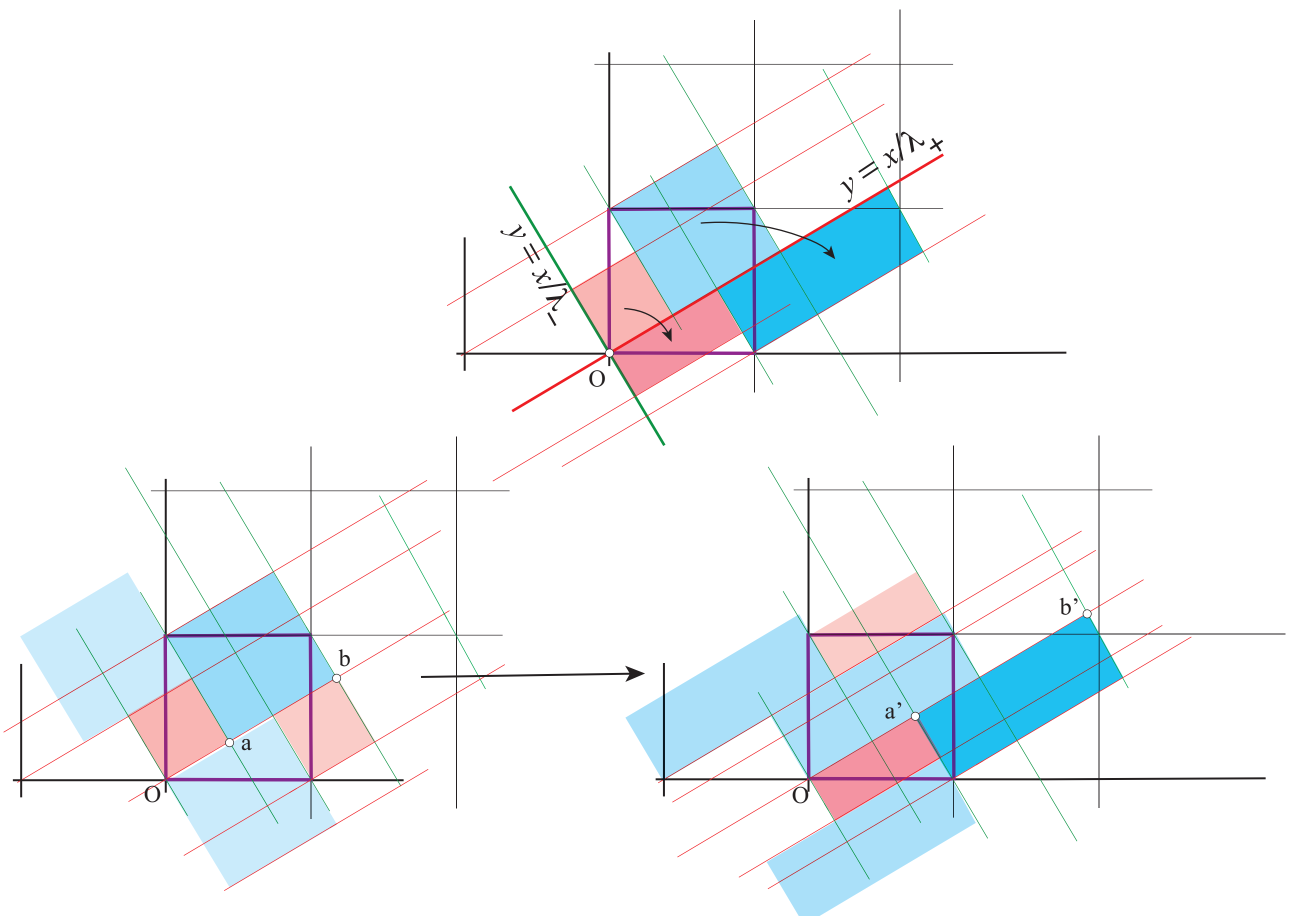

Figure H4.1: A Markov partition; the lower figures explain how to cover T^2 with the Markov partition above and its image. Red lines indicate W^u and the green W^s of the origin. They cover T^2 densely.