Homework 10 Soln

10.1 [Pedestrian Mermin-Wagner]
Let us demonstrate that for any $T > 0$ there is no long-range order (nonzero magnetization) in 2-space for systems with spin dimension $n = 2, 3, \cdots$.

Our model is given by the following Hamiltonian:

$$H = -\frac{1}{2} J \sum_{\langle i,j \rangle} s_i \cdot s_j - h \cdot \sum_i s_i,$$

(HW10.1)

where $s_i$ is the $i$th spin $\in S_{n-1}$ ($n-1$-unit sphere).

Suppose there is a long range order (i.e., nonzero magnetization). Taking $h$ as the $z$-direction, let us write the azimuthal angle of $s_i$ as $\theta_i$. We may write the Hamiltonian as

$$H = -\frac{1}{2} J \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j) - h \sum_i \cos \theta_i.$$

(HW10.2)

If the angles fluctuate only slightly from the ordered state, we may simplify the Hamiltonian as

$$H = - \int d^2 r \left[ \frac{1}{2} K (\nabla \theta)^2 + \frac{1}{2} b \theta^2 \right],$$

(HW10.3)

where $K$ and $b$ are positive constants.

(1) Find the magnetization $m = \langle \cos \theta \rangle \simeq \exp(-\langle \theta^2 \rangle / 2)$. To do this, let us be theoretical physicists. Then, the Boltzmann factor $e^{-\beta H}$ is Gaussian, so

$$\langle \theta^2 \rangle = k_B T (-K \Delta + b)^{-1} \propto \int d^2 k \frac{k_B T}{K k^2 + b},$$

(HW10.4)

where $k$ integration is from 0 to the cutoff (i.e., the inverse lattice spacing). You might think this is too much, so try to justify the above result to your satisfaction. You may ignore numerical coefficients such as $2\pi$.

Soln.

(HW10.3) may be rewritten with the aid of

$$\theta(k) = \frac{1}{(2\pi)^2} \int d^2 r e^{-i \mathbf{k} \cdot \mathbf{r}} \theta(r).$$

Then,

$$\theta(r) = \int d^2 k \ e^{i \mathbf{k} \cdot \mathbf{r}} \theta(k).$$
Putting this into (HW10.3), we obtain

\[
H = - \int d^2 r \int d^2 k \int d^2 k' \left[ \frac{1}{2} K i k \theta(k) \cdot [i k' \theta(k')] + \frac{1}{2} b [\theta(k)][\theta(k')] \right] e^{i[k+k'] \cdot r}
\]

\[
= - \int d^2 k \left[ \frac{1}{2} K i k \theta(k) \cdot [i k' \theta(k')] + \frac{1}{2} b [\theta(k)][\theta(k')] \right] (2\pi)^2 \delta(k + k')
\]

\[
= -(2\pi)^2 \int d^2 k \theta(k) \left[ \frac{1}{2} K k^2 + \frac{1}{2} b \right] \theta(-k).
\]

Therefore, \(e^{-\beta H}\) is Gaussian for \(\theta(k)\). Thus, we have

\[
\langle \theta(k)\theta(k') \rangle \propto \frac{k_B T}{K k^2 + b} \delta(k + k').
\]

Now, note that

\[
\langle \theta^2(r) \rangle = \left\langle \left[ \int d^2 k \theta(k) e^{i r \cdot k} \right]^2 \right\rangle = \int d^2 k \int d^2 k' \langle \theta(k)\theta(k') \rangle e^{i r \cdot (k + k')}
\]

\[
\propto \int d^2 k \int d^2 k' \frac{k_B T}{K k^2 + b} \delta(k + k') e^{i r \cdot (k + k')} \tag{HW10.5}
\]

\[
= \int d^2 k \frac{k_B T}{K k^2 + b} \tag{HW10.6}
\]

That is what we wished to have.

(2) Show that if the magnetic field is reduced (\(b \to 0\)), \(m\) vanishes. That is, due to fluctuation at any temperature \(m\) is zero. Thus, there is no long-range order for any model whose spin dimension is larger than 1.

**Soln.**

Let \(b = 0\). Then, writing the cutoff as \(\Lambda\), we get

\[
\int d^2 k \frac{k_B T}{K k^2} \propto \int_0^\Lambda k dk \frac{k_B T}{K k^2} = +\infty.
\]

That is, this is log-divergent. Therefore, \(m \to 0\).

10.2 [Very high temperature correlation]

Let us show that for sufficiently high temperatures the spin-spin correlation function decays exponentially:

\[
\langle \cos[\theta(r) - \theta(0)] \rangle \sim (\beta K/2)^{-|r|}. \tag{HW10.7}
\]
Here, $r = |\mathbf{r}|$ is measured in the unit of lattice spacing. The strategy to demonstrate this is to consider the propagation of the influence of the spin at the origin to the spin at $\mathbf{r}$, and we assume Markov chain-like relation:

$$\langle \cos[\theta(\mathbf{r}) - \theta(0)] \rangle \simeq \langle \cos[\theta(0) - \theta(1)] \rangle \langle \cos[\theta(1) - \theta(2)] \rangle \cdots \langle \cos[\cdots - \theta(\mathbf{r})] \rangle. \quad \text{(HW10.8)}$$

Thus, we have only to estimate $\langle \cos[\theta(0) - \theta(1)] \rangle$ between the nearest neighbor spins. Let us write $\theta(1) - \theta(0) = \delta \theta$.

(1) Show that $\langle \cos \delta \theta \rangle = \langle e^{-i\delta \theta} \rangle$.

**Soln.**

$$\langle \cos \delta \theta \rangle = \frac{1}{2} \langle e^{i\delta \theta} + e^{-i\delta \theta} \rangle.$$

The system is invariant under $\delta \theta \leftrightarrow -\delta \theta$, so we can compute one of them.

(2) The average may be computed with respect to the following density distribution function

$$P(\delta \theta) \propto e^{\beta K \cos \delta \theta} \quad \text{(HW10.9)}$$

as

$$\langle \cos \delta \theta \rangle = \int \! d\delta \theta P(\delta \theta) e^{-i\delta \theta}. \quad \text{(HW10.10)}$$

Show that

$$\langle \cos \delta \theta \rangle = \frac{\beta K}{2}. \quad \text{(HW10.11)}$$

**Hint:** Taylor-expand $P$ in powers of $K$.

**Soln.**

Following the hint, we have

$$\langle \cos \delta \theta \rangle \propto \int_0^{2\pi} d\delta \theta \ e^{\beta K \cos \delta \theta} e^{-i\delta \theta}$$

$$= \int_0^{2\pi} d\delta \theta \left[ 1 + \beta K \cos \delta \theta + \frac{1}{2} \beta^2 K^2 \cos^2 \delta \theta + \cdots \right] e^{-i\delta \theta}.$$

Clearly the terms with different periodicities vanish. Thus

$$\langle \cos \delta \theta \rangle \propto \int_0^{2\pi} d\delta \theta \ \beta K \cos \delta \theta e^{-i\delta \theta} = \beta K \pi.$$

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1 A slightly more detailed calculation easily tells us other terms do not contribute, if $\beta K$ is sufficiently small.
We must compute the normalization constant:

\[ \int d\delta \theta e^{-\beta K \cos \delta \theta} = 2\pi. \]

Therefore,

\[ \langle \cos \delta \theta \rangle = \frac{\beta K \pi}{2\pi} = \frac{\beta K}{2}. \]

We may conclude (HW10.7). That is, the spin-spin correlation function decays exponentially. The correlation length reads

\[ \xi = \frac{1}{\log(2/\beta K)} \sim \frac{1}{\log T}. \]

10.3 [Very low temperature correlation]
At very low temperatures, \( \theta \) changes mostly smoothly, so we may use (HW10.3).

1. Show

\[ \langle \cos[\theta(r) - \theta(0)] \rangle = e^{-[g(0)-g(r)]}, \quad (\text{HW10.12}) \]

where

\[ g(r) = \langle \theta(r)\theta(0) \rangle. \quad (\text{HW10.13}) \]

Hint: \( \theta \) is Gaussian, so you need only up to the second cumulant.

SoIn.

\[
\langle \cos[\theta(r) - \theta(0)] \rangle = \text{Re} \langle e^{i[\theta(r)-\theta(0)]} \rangle \\
= \text{Re} e^{i(\theta(r)-\theta(0))-\frac{1}{2}[\theta(r)-\theta(0)]^2} \\
= \text{Re} e^{-\frac{1}{2}[\theta(r)-\theta(0)]^2}.
\]

Note that this is exact. Now,

\[
\langle [\theta(r) - \theta(0)]^2 \rangle = \langle \theta(r)^2 - 2\theta(r)\theta(0) + \theta(0)^2 \rangle \\
= 2\langle \theta^2 \rangle - 2\langle \theta(r)\theta(0) \rangle = 2[g(0) - g(r)].
\]

Thus, we have obtained (HW10.12).

From (HW10.4) you may be able to guess the following formula:

\[
g(r) = \frac{k_B T}{4\pi^2 K} \int dk \frac{1}{k^2} e^{-ik \cdot r}. \quad (\text{HW10.14})
\]
Using

\[ \frac{1}{2\pi} \int d\theta e^{ikr\cos \theta} = J_0(kr), \quad (\text{HW10.15}) \]

where \( J_0 \) is the Bessel function, obtain

\[ g(0) - g(r) = \frac{k_B T}{2\pi K} \int_{\Lambda} dk \frac{1}{k} (1 - J_0(kr)) = \frac{k_B T}{2\pi K} \int_{\Lambda} dy \frac{1}{y} (1 - J_0(y)). \quad (\text{HW10.16}) \]

(2) Obtain

\[ \langle \cos(\theta(r) - \theta(0)) \rangle = \left( \frac{1}{r\Lambda} \right)^{k_B T/2\pi K}. \quad (\text{HW10.17}) \]

This is not exponential. Thus, we may conclude that there is a \textit{quasi long-range order}. Since the high temperature phase and the low temperature phase are qualitatively different, there must be at least one phase transition along the temperature axis.

Hint: Show that the integral is well defined near \( k = 0 \). For very large \( r \), we may ignore \( J_0 \).

\textbf{Soln.}

\( J_0(0) = 1 \) and \( J_0(x) \) is holomorphic around \( x = 0 \), so the integral converges. We are interested in the large scale behavior of the correlation, so we have only to discuss the large \( r \) behavior. In this limit \( J_0 \) vanishes, so we need only the log integral:

\[ g(0) - g(r) \simeq \frac{k_B T}{2\pi K} \int_{\Lambda} dy \frac{1}{y} = \frac{k_B T}{2\pi K} \log(r\Lambda) + \text{const}, \quad (\text{HW10.18}) \]

where the ‘const’ comes from the non-asymptotic terms. Therefore,

\[ \langle \cos[\theta(r) - \theta(0)] \rangle = e^{-[g(0)-g(r)]} \propto e^{-(k_B T/2\pi K)\log(r\Lambda)} \propto \left( \frac{1}{r} \right)^{k_B T/2\pi K}. \quad (\text{HW10.19}) \]

\textbf{Remark}. Since \( m = \langle \cos \theta(r) \rangle \), you might think the magnetization is a local quantity, but this formula assumes the translational symmetry, so non-zero \( m \) means globally everywhere \( \cos \theta \) or \( \theta \) is uniform on the average. Thus, when \( m \neq 0 \) we can unambiguously claim that the phase has a long-range order. In this sense, the XY model in 2-space does not show any long range order. However, as you have seen, the correlation function changes drastically at some temperature, below which the (correlation) exponent \( \nu = k_B T/2\pi K \) changes gradually. Thus, the system is always at criticality below this temperature. The story beyond this point requires an RG. See the XY.pdf posted near this solution.

\footnote{You must know common sense knowledge of Bessel functions.}