Homework 1

Due on Feb 7 (W by midnight),\(^1\) 2018 to sblkrsh2@illinois.edu.
If you use TeX (and pdf), you can submit your draft to me for comments if before 18:00 of Feb 3 (Sat), 2018. This time the homework is easy, so probably most of you do not need any help.
The source file of this homework is posted on our course page.

1.1 [Density matrix and entanglement entropy]
Consider a system consisting of two (electron) spins A and B. Suppose the state is in a (pure) state described by

\[ |\psi\rangle = \alpha |0\rangle_A |1\rangle_B + \beta |1\rangle_A |0\rangle_B, \]  

(0.0.1)

where \(|\alpha|^2 + |\beta|^2 = 1\) (normalized), and \(|s + 1/2\rangle_X\) describes the spin state \(s (= \pm 1/2\) spin state) of ‘electron’ X.

(1) Write down the density operator \(\rho\) for the whole system (a trivial question), and check \(\rho^2 = \rho\).

(2) Compute the partial density operator \(\rho_A\) for electron A by tracing out the spin states for B (i.e., take the partial trace for the states of spin B: \(\rho_A = \text{Tr}_B \rho\)).

The von Neumann entropy \(S\) of the state whose density operator is \(\rho\) is defined as

\[ S = -\text{Tr} \rho \log \rho. \]  

(0.0.2)

Generally speaking, as we will learn later, entropy measures the information hidden from the observer (or measures how many different (micro)states in the state that the observer regards as a single state).

(3) Show that for any pure state \(S = 0\). Thus, we know the entropy for the whole system in our case is zero.
This may be interpreted as: there is nothing we can know further about the system, if we know its (microscopic) state.

Tracing out spin B implies that we do not observe B. Then, the increase of the von Neumann entropy due to this tracing (in our case the von Neumann entropy of \(\rho_A\)) measures the amount of information we have lost by ignoring the relation between A and B. This relation is due to entanglement, so this entropy increase measures the strength of quantum entanglement between A and B.

(4) Compute the entanglement entropy for this system (i.e., compute \(S_A = S(\rho_A)\)). When is this entropy maximized? (Change \(\alpha\)).

(5) If \(\alpha = 0\), then \(S_A = 0\). What is the interpretation of the result? [trivial]

1.2 In 1.1 we ‘shaved off’ a part of the system. This procedure converted a pure state into a mixed state. Can we combine a system A in a mixed state \(\rho_A\) and some other system B to realize a pure state for the combined system? That is, can we make \(\rho = |\psi\rangle\langle\psi|\) such that \(|\psi\rangle = \sum \beta_{ij} |a_i\rangle |b_j\rangle\) and \(\text{Tr} \rho = \rho_A\) for a given \(\rho_A\), where \(\{ |a_i\rangle \}\) is an ON basis for A and \(\{ |b_j\rangle \}\) that for B? The answer is affirmative, and is called the purification theorem. Let us demonstrate this easy (but extremely useful) theorem.\(^2\)

\(^1\)The homework cycle will be Tu to W, but hopefully, the problems will be posted by Monday Noon.
\(^2\)Because before interaction with the system we are interested in, we may assume the whole system to be in a pure state.
(1) Show that $\rho_A$ can always be written in the following form: $\rho_A = \sum |a_i\rangle p_i \langle a_i|$, where \{|a_i\}\} is an ON basis for system A and $\sum p_i = 1$, $p_i \geq 0$. [trivial]

(2) A possible pure state is

$$|\psi\rangle = \sum_i \sqrt{p_i} |a_i\rangle |b_i\rangle.$$  \hspace{1cm} (0.0.3)

Check that this gives $\text{Tr}_B |\psi\rangle \langle \psi| = \rho_A$. [trivial]

1.3 Two systems A and B have their own vector spaces (Hilbert spaces). Then, for any pure state $|\psi\rangle$ for the combined system of A and B, there are ON basis \{|a_i\}\} for A and that \{|b_j\}\} for B such that

$$|\psi\rangle = \sum_i \sqrt{p_i} |a_i\rangle |b_i\rangle,$$  \hspace{1cm} (0.0.4)

where $\sum p_i = 1$ ($p_i \geq 0$). This is called the \textit{Schmidt decomposition}. Give an elegant proof.\footnote{You can look up any reference, say Wikipedia; if you think what you find is elegant enough, you can simply submit it.}
