Homework 3
due Feb 15, 2017 (at 18:00) [in PDF with LaTeX to Tong Wang twang92@illinois.edu]

As I announced, HWs are regarded as substitutes for Discussion, so you may submit your ‘draft’ electronically for my comments/suggestions/hints (by the time reasonably before the deadline). Also you can freely discuss with your friends as long as it facilitates your understanding. I hope all of you will get 100%.

1. In 1907 Einstein remarked: for a Brownian particle its average velocity may be obtained from its short-time displacement, but a very short time average does not make sense (as the average over the true velocity).

   It is extremely hard (or almost impossible) to verify the equipartition of energy for a Brownian particle, even though Langevin simply declared the average in (9.4) to be $3k_BT/2$. This was another reason (according to some people) that Brownian motion was not seriously discussed by thermal physicists.

   The mean displacement during time $t$ is proportional to $\sqrt{t}$. This implies that the instantaneous velocity must be undefinable; the motion is very erratic. Perrin commented that the Brownian path is nowhere differentiable.

   A Brownian particle of radius 0.1 $\mu$m is suspended in water (say, at 300 K). Estimate the time needed for the initial speed to decay to its 1/10. What does this tell you about the observability of the speed of the particle? It is clear that you must have a time resolution at least one order better than this to confirm the equipartition law. [This is a ‘real physics’ question, so you must assume the mass of the Brownian particle you discuss, etc.]

2. How long will it take for a complete ribosome to diffuse across a eukaryotic cell or a bacterial cell? Again this is a ‘real science’ question, so you must look for the rough size of the ribosome complex, cell size, etc. (however, let us ignore the so-called macromolecular crowding effect). You must choose reasonable values. In contrast to eukaryotic cells, bacterial cells lack molecular transporters such as kinesins. Can you comment on this general observation, referring to your quantitative estimates?

3. Let us assume a free-floating Brownian particle obeys the following Newton’s equation of motion

   $\frac{m}{m} \frac{dv}{dt} = -\zeta v + w(t), \quad (0.0.1)$

   where $w(t)$ is a stationary random force whose time correlation reads

   $\langle w(t)w^T(s) \rangle = 2\alpha I \delta(t - s). \quad (0.0.2)$

   Here, $I$ is the $3 \times 3$ unit matrix and $\alpha$ is a positive constant to be determined.

   (1) Assuming the initial velocity to be $v(0)$, solve the above equation of motion for $v(t)$ in terms of $w(s) (s \in [0, t])$ and $v(0)$.

   (2) Using the result in (1), compute the mean square velocity $\langle v(t)^2 \rangle$. Throughout 3 the average is an ensemble average over many identical samples.

   (3) Assuming that the system is in equilibrium (i.e., we observe the Brownian particle suspended
in a fluid in equilibrium at temperature $T$; you may assume the equipartition of kinetic energy), determine $\alpha$. Thus, you have fixed the noise amplitude in terms of the friction constant $\zeta$ and $T$ (the fluctuation-dissipation relation in this case).

(4) Compute the velocity autocorrelation function $\langle \mathbf{v}(t) \cdot \mathbf{v}(s) \rangle$, assuming the system is in equilibrium.

(5) Show the diffusion constant of the Brownian particle reads (the Green-Kubo relation for the diffusion constant)

$$D = \frac{1}{3} \int_0^\infty \langle \mathbf{v}(s) \cdot \mathbf{v}(0) \rangle ds.$$   \hspace{1cm} (0.0.3)

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4 [Modern Perrin]
Using particles of radius $a = 0.550 \, \mu m$ suspended in water at $T = 23.6^\circ C$ (shear viscosity $\eta = 0.919 \, \text{mPa} \cdot \text{s}$), the mean square displacements were measured. The results on a 2D observation table are summarized in the following figure.\(^1\) The black line indicates the average behavior.

![ MSD vs Time ]

Assuming that we know $R = 8.31 \, \text{J/mol} \cdot \text{K}$, estimate Avogadro's constant $N_A$.

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\(^1\)taken from Marco A. Catipovic, Paul M. Tyler, Josef G. Trapani, and Ashley R. Carter, Improving the quantification of Brownian motion, Am. J. Phys. 81 435 (2013).