Homework 10

due April 26, 2017 (at 18:00) [in PDF with LaTeX to Tong Wang twang92@illinois.edu]

As I announced, HWs are regarded as substitutes for Discussion, so you may submit your ‘draft’ electronically for my comments/suggestions/hints (by the time reasonably before the deadline). Also you can freely discuss with your friends as long as it facilitates your understanding. I hope all of you will get 100%.

1 Let us consider the 2-Ising model on a square lattice using the decimation technique: we average over the spins on the \((i, j)\) lattice with \(i + j\) even (i.e., checker-pattern summation; sum over the spins on the black lattice points in the following figure).

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Left arrow indicates coarse-graining (decimation) \(K\); Right arrow indicates spatial scaling (shrinking with rate \(\ell\)) \(S\) to maintain the bond length; \(SK\) is the renormalization transformation \(R\) in this case.

As you can check easily, this procedure ‘re’produces not only the nearest neighbor but also next nearest neighbor spin-spin interactions (and actually much more). Therefore, let us span the space of the Hamiltonians in the following form

\[
H = J \sum_{nn} s_i s_j + K \sum_{nnn} s_i s_j, \quad (HW10.1)
\]

where \(nn\) means the summation over all the nearest-neighbor pairs, and \(nnn\) over the next-nearest-neighbor pairs.

The partition function may be understood as the sum over spins \((s's)\) on the white lattice of the
product of the squares (see the above figure) summed over the center spin values ($\pm 1$):

$$Z = \sum_{\{s=\pm 1\}} \prod \left( \sum_{\sigma_0=\pm 1} e^{J(s_1+s_2+s_3+s_4)\sigma_0+K(s_1s_2+s_2s_3+s_3s_4+s_4s_1)} \right); \quad \text{(HW10.2)}$$

The quantity in the round parentheses in (HW10.2) corresponds to the sum over $\sigma_0$ in the white square in the figure.

(1) First, an extremely easy question. Compute

$$\Sigma_1 = \sum_{\sigma_0=\pm 1} e^{J(s_1+s_2+s_3+s_4)\sigma_0+K(s_1s_2+s_2s_3+s_3s_4+s_4s_1)}. \quad \text{(HW10.3)}$$

We wish to recast the partition function in the following form (times some constant):

$$Z = \sum_{\{s=\pm 1\}} \exp \left( J' \sum_{nn} s_is_j + K' \sum_{nnn} s_is_j \right). \quad \text{(HW10.4)}$$

This implies to equate $\Sigma_1$ and the following term (times some constant)

$$e^{J'(s_1s_2+s_2s_3+s_3s_4+s_4s_1)+K'(s_1s_3+s_2s_4)}. \quad \text{(HW10.5)}$$

That is,

$$\text{Result of (1)} = A e^{J'(s_1s_2+s_2s_3+s_3s_4+s_4s_1)+K'(s_1s_3+s_2s_4)}. \quad \text{(HW10.6)}$$

Here $A$ is a numerical factor independent of $J$ and $K$. This should hold for $s_i = \pm 1$ for $i = 1, 2, 3, 4$.

If we take the spin up-down symmetry of the system, there are 4 distinct cases we must consider: (A) all up, (B) one down, and two downs (C) with downs being next to each other or (D) not (see the figure below). However, we must determine only three unknowns $A, J'$ and $K'$, so, generally speaking, a fully consistent relation is impossible. We must introduce some approximation.

(2) The most simple-minded approach is to expand the both sides of (HW10.6) in powers of coefficients $(J, K, J', K')$ and compare the powers of $s$'s for lowest needed order terms. However, there is one complication: for example, $s_1s_2$ interaction is created not only from the white square but also from the one in its ‘north-west’ neighborhood. Such contributions just double the contribution from the white square. Thus, we get $J' = 2J^2 + K$ (2 is due to the complication just mentioned). Find one more relation giving $K'$. [Don’t forget $s^2 = 1$. Instead of naive expansion, to compare log of $\Sigma_1$ and ( ) in (HW10.4) is recommended; do not forget the complication explained to get the factor 2 mentioned above.]

(3) There are three fixed points of the map $T : (J, K) \rightarrow (J', K')$, $F_0 = (0, 0)$, $F_\infty = (\infty, \infty)$ and
the finite one $F$ (called the nontrivial fixed point). $F_0$ and $F_\infty$ are stable fixed points (sinks). Find $F$ and explain what states these three fixed points correspond to.

(4) Linearize the map $T$ around the fixed point $F$, and find its eigenvalues.

(5) Unfortunately, the eigenvalue whose absolute value is less than 1 (irrelevant direction) is negative due to our poor approximation, but still the larger eigenvalue is larger than 1, which corresponds to $\ell^{y_1}$. Find $y_1$. [What is $\ell$ in our case?]

The following questions are about the interpretation of $y_1$.

(6) This $y_1$ is the exponent appearing in the Kadanoff construction. This is directly related to the correlation length $\xi$, which diverges as $\xi \sim |\tau|^{-\nu}$. $\nu$ is the critical exponent for the correlation length. Since $\xi$ is a function of $\tau = (T - T_c)/T_c$ when there is no magnetic field, we can write

$$\xi = \Xi(\tau).$$

(HW10.7) What is the scaling relation for this? That is, if we look at the system from the distance $\ell$ times as large as the original observation distance, how does (HW10.7) depend on $\ell$ according to Kadanoff’s idea? Then, show $\nu = 1/y_1$.

(7) Suppose the magnetic system is in a finite box of linear scale $L$. As long as $L \gg \xi$, the system does not feel the finiteness of the box, but if $\xi$ becomes comparable to $L$, large scale fluctuations are suppressed by the boundary. Therefore, the susceptibility of a finite magnet must be dependent not only on $\tau$ but also on $L$ (let us assume there is no magnetic field):

$$\chi = X(\tau, L).$$

(HW10.8) Write down the scaling relation for this, introducing the scaling factor $\ell$ as usual (i.e., apply the Kadanoff construction) [Look at (33.18) and (33.20)]. Then eliminate $\ell$ as usual, using $\tau\ell^{y_1} = 1$, assuming $\tau > 0$. [Since $\nu = 1/y_1$, note that we can find $y_2$ in terms of the critical exponent $\gamma$ and $\nu$.

(8) Since the critical divergence is suppressed by the walls, $\chi$ must stay finite at any temperature. Still near the $T_c$ for a very large system, $\chi$ should become large. How does its peak height depend on $L$? The $\chi$ peak height is some power of $L$. Find the power in terms of the critical exponents we already know ($\alpha$, $\beta$, $\gamma$ and/or $\nu$)? [The key point is that $\chi$ obtained in (7) as a function of $\tau$ and $L$ should not diverge, even if $\tau \to 0$, so $\tau$ dependence must be eliminated from the expression of $\chi$. This requirement fixes the power of $L$.]