3.4 Two approaches to renormalization

The phenomenology has, roughly speaking, the structure: ‘phenomenology’ = ‘universal structure’ + ‘phenomenological parameters.’

The universal structure (if any) should be found through searching for results that remain invariant under changing microscopic details, that is, through pursuing ‘structurally stable’ results by modifying the ‘unobservable details.’ There are at least two strategies to pursue invariance under microscopic changes.

1. Observe the system from increasing distance with a fixed resolution (resolving power) and find the properties that persist to be observed.
2. Find the structure insensitive to the change of microscopic parameters (a strategy faithful to the idea explained in the preceding section).

The renormalization strategy based on (1) is called the *Wilson-Kadanoff renormalization procedure* and that based on (2) is called the *Stückelberg-Petermann renormalization procedure*.

Suppose we wish to make a phenomenology for an observable (say, the heat capacity near the critical point of a magnet).

To realize (2) we need a fairly explicit form of the quantity as a function of microscopic parameters, so (approximate) methods to compute it is required (e.g., perturbation calculation). Often minimal models are studied.

The approach (1) is equivalent to coarse-graining the system while shrinking its size. We could perform this transformation (*renormalization transformation*) step by step to watch how observables change. If the system is sufficiently large and if its boundaries need not be taken into account, then the global features must keep being observed. On the other hand, small scale details that have no global effects would become increasingly invisible by the procedure. Therefore, the properties that survive the numerous applications of the renormalization transformation (= coarse-graining + shrinking) must govern the stable structure we are pursuing.

Against the research program advocated in this Chapter (and in the next Chapter that carries on its philosophy) there will be harsh criticism from the *ab initio* side people who assert everything should be obtained from *ab initio* quantum mechanical calculations. What we wish to understand is, however, the question such as: why are both a piece of chalk and flour white? The answer to this question should not require any details of atoms and molecules. The author does not claim that the atomistic approach is wrong, but it is a

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59 These details need not be unobservable; they could be the factors we do not wish to consider, although whether we are allowed to ignore them or not cannot be up to us to decide.

60 Even if one admits that material consists of atoms and molecules, the assertion that they simultaneously obey mechanics is a transcendental assertion. Of course, there is a (majority) opinion that even if this assertion cannot be directly verified, if the idea always works well, we may conclude that the assertion is
simple fact that there are many phenomena for which pursuit of microscopic details does not lead us to their understanding.\textsuperscript{61} The idea of the contemporary computational physics is reductionism through and through; if you go to the ultimate microscopic description of a system, you would get the truth, nothing but only the truth. The reason why this strategy is often regarded as the frontier of science is that we do not know what is important and what not for most phenomena. We do not know where we need not pay close attention. Rigorous solutions are needed in mathematics if we do not have much insight into the problem. Or, rigorous proofs are needed because our “math sense” is feeble.\textsuperscript{62} Therefore, our goal must be that we can have insight without any unnecessary computation and must not be that we can perform all the computations.

### 3.5 ABC of renormalization

The purpose of this section is to use supposedly the simplest example to have a hand-on experience of renormalization methods. The example, though simple, exhausts all the key points.

Self-similar geometrical objects like the von Koch curve (Fig. 3.5.1)\textsuperscript{63} appear often in simple systems without representative scales. For example, for chaos, the microscopic scale is expanded indefinitely, so there is no small fundamental scale. If a trajectory cannot cover a region densely, it is often the case that we have such a self-similar structure transversal to the orbits (as in the Lorenz system in Section 2.1). The system producing chaos is simple, so repetition occurs.

The length of the von Koch curve depends on how we measure it (at what scale we observe the curve). Our problem is a simplified version of how to measure the length of a coastal line. Such self-similar curves are often called fractal curves.\textsuperscript{64} The problem of measuring correct. However, when it is asserted that something always works, it is not rare that ‘always’ is actually due to biased sampling; it is often the case that we see only things we wish to see. Where there is a logical hole, there is very often a real hole.

\textsuperscript{61}Here, ‘whiteness’ is a metaphor of an interesting property we wish to understand. There can be an opinion that atoms and molecules look irrelevant simply because we pay attention to such an uninteresting secondary property. However, the existence of highly nontrivial phenomenologies almost free from materials details (as illustrated in Section 3.1) should be enough to silence this opinion.

\textsuperscript{62}according to Kunihiko Kodaira

\textsuperscript{63}von Koch (1870-1924) is a mathematician who proved that, if the Riemann hypothesis holds, the discrepancy between the true distribution of prime numbers (not exceeding $X$) and the Gauss’s distribution $\int_2^X dx/\log x$ is bounded by a number proportional to $\sqrt{X}\log X$

\textsuperscript{64}About fractals, see the book by the originator of the concept; B. B. Mandelbrot, \textit{Fractal Geometry of Nature} (W. H. Freeman, San Francisco, 1983). If attractors of chaotic systems are visualized with the aid of CG, numerous fractal-like figures can be seen. An unbiased first impression would be that there is no such rather monotonous repetitive patterns are in Nature. Those who are impressed by CG fractals are probably the people who have never been overwhelmed by the diversity and complexity of a forest. One must feel the distinction between ‘complex systems’ and not complex systems as fractal curves.

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Figure 3.5.1: How to construct the von Koch curve. First, take a line segment of length \( W \) and make a segment of length \( 1/3 \) of the former. Prepare four such segments and make a piecewise linear figure with a triangular mound at the center. This is Step 1. Next, each linear segment of this figure (a representative is encircled with a broken ellipse) is replaced by a \( 1/3 \) shrunk copy of the whole figure. This is Step 2. The resultant figure at the bottom of the figure is made of unit segments (monomer units) of length \( \ell = W/9 \). These length \( W/9 \) monomer units are all replaced by the small copy of the figure constructed in Step 1 as shown just below Step 3. After Step 3 the length of the ‘monomer unit’ is \( W/3^3 \). Now, we repeat this procedure \textit{ad infinitum} beyond Step 4, and we will obtain the von Koch curve. In this section, we regard the length of the ‘monomer unit’ to be finite, so we repeat these steps only finite times. However, since the number of repetition \( n \) is large, the length of the monomer unit \( \ell = W/3^n \) is invisibly small.

Measuring the coast length has a deeper meaning than is understood by the fractal aficionados, because the length is always fluctuating due to waves, and in the limit of accuracy, the coast itself becomes meaningless (non-existent). This illustrates that the existence of objects in the microscopic world and that at our scale can be fairly different.

Let \( \ell \) be the length of the smallest unit of the curve (‘the atom size’), \( L \) be the total length of the curve, and \( W \) be, as in Fig. 3.5.1, the span of the curve. From these lengths, we can make two independent dimensionless quantities: \( L/W \) and \( W/\ell \), so dimensionally analytically (see Appendix 3.5A) we can conclude

\[
\frac{L}{W} = f\left(\frac{W}{\ell}\right),
\]  

(3.5.1)

where \( f \) is an appropriate function. As is explained in Appendix 3.5A, this is required by the invariance of the relation under changing the unit of length. According to the ordinary common-sense dimensional analysis instruction, we may ignore the dimensionless quantities whose values are too large or too close to zero. If \( f \) in (3.5.1) does not diverge in the \( \ell \to 0 \) limit, then \( L \propto W \). That is, the conventional instruction works. For example, the circumference \( L \) of the regular \( n \)-gon with the edge length of \( \ell \) inscribed in the circle of diameter \( W \) reads

\[
L = \frac{\pi \ell}{\text{Arcsin}(\ell/W)} = W \frac{\pi(\ell/W)}{\text{Arcsin}(\ell/W)}.
\]  

(3.5.2)

Therefore, indeed this has a structure of (3.5.1). Furthermore, \( f(x) = \pi/[x\text{Arcsin}(1/x)] \) converges to \( \pi \) in the \( x \to \infty \) limit. Thus, the ordinary instruction is very reasonable that recommends to ignore \( W/\ell \) if it is extremely large.

However, for the von Koch curve, we can easily see that this is not the case. Although the ‘hidden length’ \( \ell \) is disparate from \( W \), we cannot ignore it (there is more systematic
explanation in Appendix 3.5B). This is exactly the scenario of interference between disparate scales. The nonlinearity in this example is in the relation between the number of units (at a certain scale) and the total span of the curve $W$. However, we will see that the effect of this hidden scale (or the effect of the divergence in the $\ell \to 0$ limit) does not show up in the result haphazardly. That is, the length of the von Koch curve has the structure of phenomenology.

Let us check the above assertion beforehand without using renormalization. To this end $f$ is determined in an elementary fashion. Applying $n$-times the procedure explained in Fig. 3.5.1 to construct the von Koch curve, we realize that the monomer unit length is $\ell = W/3^n$, and that the total length of the curve is $L = \left(\frac{4}{3}\right)^n W$. Since

$$n = \log_3(W/\ell),$$

we obtain

$$L = W^{\log 4/\log 3} \ell^{1-\log 4/\log 3}.$$  (3.5.4)

That is, $f(x) = x^{\log 4/\log 3 - 1}$. If we collect von Koch curves with various values of $\ell$ (i.e., the curves with different microscopic structures), their ‘true’ length $L$ as a function of $W$ is always proportional to $W^{\log 4/\log 3}$ (the universal mathematical structure). However, the proportionality constant between $W^{\log 4/\log 3}$ and $L$ varies (phenomenological adjustable parameter), since it is sensitive to $\ell$; although $\ell$ varies, for each example of the von Koch curve, it is a definite constant. Notice that the ‘phenomenological parameter’ $\ell^{1-\log 4/\log 3}$ diverges in the $\ell \to 0$ limit. This is a trivial example, but the structure of phenomenology stated in Section 3.1 appears clearly in the result. Although the value of the phenomenological parameter depends on the detail $\ell$, notice that the functional form $\ell^{1-\log 4/\log 3}$ is universal to all the von Koch curves. If we pay attention only to macroscopically easily observable $L \propto W^{\log 4/\log 3}$, since both $L$ and $W$ have the dimension of length, the result seems to suggest that the dimension of $L$ deviates from the simple dimensional analytical result $L \propto W$ by $\log 4/\log 3 - 1$. This deviation is called the anomalous dimension. This is a telltale sign of a hidden dimensional quantity that cannot be ignored.

Let us first perform the strategy (1) the Wilson-Kadanoff RG: “To observe the system far away with a constant eyesight.” The each step of this procedure is:

(i) Perform coarse-graining (this procedure is called the Kadanoff transformation $K_{65}$); During this procedure, the size of the figure is intact: $W = K(W)$. However, the size of the smallest unit (the monomer unit) becomes larger: $\ell \to K(\ell) = 3\ell$. Therefore, the total length of the curve becomes $L \to K(L) = (3/4)L$. Next,

65What is the reason for Professor Kadanoff has not been awarded a Nobel Prize? It is very likely because the judges are not truly creative people. It was Kadanoff who saw the mathematical core of the procedure. There are people who say that he should have done one more step (at least in retrospect), but if one is convinced (perhaps unconsciously) that he grasps the crux of the matter, it is all too natural to relax at that point. Checkmate has already been achieved. In these days actually demonstrating every step of checkmate through the end seems excessively appreciated (because everyone can understand what is going on). This is amply proved by the appallingy late Nobel Prize for Y. Nambu.

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(ii) Apply a scaling transformation $S$ that shrinks the whole figure uniformly so that the size of the smallest unit returns to that of the original figure (scale the whole figure so that the monomer size becomes the same as the original figure): $S(\ell) = \ell/3$ (indeed, $S(K(\ell)) = \ell$). Obviously, $S(L) = L/3$ and $S(W) = W/3$ hold.

The composition of the above two transformations $R \equiv SK$ is defined as the renormalization transformation. By construction, obviously, it preserves the microscopic structure of the von Koch curve. $R^2$ is also a renormalization transformation, so we say the totality of such transformations makes a renormalization group (RG). $R$ is also called a renormalization group transformation. Since $R(L) = L/4$, we obtain

$$R^n(L) \equiv L(n) = L/4^n. \tag{3.5.5}$$

Here, $R(W) = W/3$. To count the number of renormalization transformation applied, let us introduce the dilation parameter $n$ with the aid of $W(n) = R^n(W) = W/3^n$ as $n = \ln\{W/W(n)\}/\ln 3$. Then, (3.5.5) may be rewritten as

$$L = 4^nL(n) = \left(\frac{W}{W(n)}\right)^{\ln 4/\ln 3} L(n). \tag{3.5.6}$$

If we choose $n$ appropriately to make $L(n)$ and $W(n)$ of order 1, we obtain (3.5.4).

Next, let us discuss the procedure (2) the Stückelberg-Petermann RG procedure: “To pursue invariants under changing microscopic details.” The basic idea is: shake the system

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66 Actually, the inverse transformation is not defined, so it is not a group. However, it has the unit element (that corresponds to doing nothing), correctly speaking the totality of these transformations makes a monoid which is slightly more structured than a semigroup.
and remove the most shaken parts, and then the rest should be universal. In the current example \( L/\ell^{1-\log 4/\log 3} \) is the invariant that is not shaken even if \( \ell \) is shaken. That is, if we could find such a quantity, we can separate out the universal structure in the phenomenological description.

The quantities a macroscopic observer knows about the von Koch curve are \( W \), the scale of observation (resolution) \( \lambda \) and the length \( \tilde{L} \) measured at this scale. The true length \( L \) is not observable, but this and \( \tilde{L} \) should be proportional (if the observation scale is fixed):

\[
\tilde{L} = Z L. \tag{3.5.7}
\]

This may be called “the condition of realism.” If this relation is not assumed, \( \tilde{L} \) does not tell us anything about the object.\(^{67}\) The divergence in the limit of small \( \ell \to 0 \) (the ‘smallness’ of a quantity is meaningless unless it is compared with something else, so, correctly speaking, in the limit of \( \ell/\lambda \to 0 \)) cannot be observed as long as our resolution is \( \lambda \). Consequently, \( Z \) must cancel the divergence in \( L \) so that it cannot be seen in \( \tilde{L} \). Such a constant that absorbs divergence is called a renormalization constant. If \( \ell \) is shrunk by \( 1/3 \) (i.e., to \( \ell/3 \)), the total length of the curve is multiplied by \( 4/3 \), so the divergence of \( L \) in the \( \ell \to 0 \) limit should behave as \( (4/3)^{-\log 4/\log 3} \).\(^{68}\) Therefore, the renormalization constant \( Z \) should be selected to remove this divergence \( \ell^{1-\log 4/\log 3} \) (i.e., \( ZL \) does not have any divergence): \( Z(\lambda/\ell) \propto (\lambda/\ell)^{1-\log 4/\log 3} \). \( Z \) is dimensionless, so it must be a function of the dimensionless quantity \( \ell/\lambda \) (if \( L/\ell \) is chosen instead, we lose the proportionality between \( L \) and \( \tilde{L} \), so no other choice is possible.)

\( \lambda \) is the quantity the observer has introduced, and has nothing to do with the system (the von Koch curve) itself. Therefore, the ‘true’ length \( L \) cannot depend on \( \lambda \) (this is a consequence of our belief that the external world exists).\(^{69}\) That is, if a system (model) is fixed (i.e., \( \ell \) and \( W \) are fixed), then even if \( \lambda \) is changed, \( L \) cannot change:

\[
\lambda \frac{\partial L}{\partial \lambda} = 0. \tag{3.5.8}
\]

Here, the multiplying extra \( \lambda \) has no deep meaning; it is only to make the subsequent formulas streamlined. Its essence is \( \partial L/\partial \lambda = 0 \): “The world exists irrespective of our existence.”\(^{70}\)

\(^{67}\)Needless to say, things we can never observe may be assumed to be non-existent, and it is conceivable to assume only \( \tilde{L} \) is real. However, this is not the way we do physics at present.

\(^{68}\)If we let \( L = g(\ell) \), \( g(x/3) = (4/3)g(x) \) is required. If we assume \( g(x) \propto x^\alpha \), \( \alpha \) satisfies \((1/3)^\alpha = 4/3\). This functional equation has many different solutions, but it is not hard to guess this simple form.

The reader might have felt strange, looking at the result in the text: a dimension quantity becomes the variable of the logarithmic function. Correctly speaking, \( \ell \) there must be \( \ell/\lambda \).

\(^{69}\)Needless to say, no one can prove the existence of the world beyond our cognition. The belief here is, precisely speaking, the belief in the notion that there is no contradiction due to assuming the existence of the external world. Phenomenologically speaking (or if Husserl’s Ideas of Phenomenology is mimicked), it is the belief in the notion that our cognition certainly imparted to us can hit what is not given to us or in the notion that there is a reason for the undoubtedness of the external world.

\(^{70}\)But, as discussed in a previous footnote, the microscopic world need not exist definitively in the macroscopic sense. Just as Monet’s waterlily, there need not be any internal structure when closely observed. The meaning of ‘existence’ is subtle, and need not be the same as our daily sense.
This condition can also be said as ‘the condition of objectivity.’

On the other hand, the quantity the macroscopic observer knows are $W$, the actually observed $\tilde{L}$, and the observation scale $\lambda$, so the result of her dimensional analysis reads:

$$\frac{\tilde{L}}{\lambda} = f\left(\frac{W}{\lambda}\right). \quad (3.5.9)$$

$f$ here is also a well-behaved function. From (3.5.7) and this, we have

$$L = Z^{-1}\lambda f\left(\frac{W}{\lambda}\right). \quad (3.5.10)$$

Introducing this into (3.5.8), we obtain (it is wise to compute the logarithmic derivative $\partial \log L/\partial \log \lambda$)

$$f(x) - \alpha f(x) - xf'(x) = 0, \quad (3.5.11)$$

where

$$\alpha \equiv \partial \log Z/\partial \log \lambda. \quad (3.5.12)$$

Such equations as (3.5.8) or its consequence (3.5.11) are called renormalization group (RG) equations. Especially if $\alpha$ converges to a constant in the $\ell \to 0$ limit (for the current example, it of course converges to $\alpha = 1 - \log 4/\log 3$), (3.5.11) becomes an equation governing the universal relation independent of the microdetails.

Solving (3.5.11), we obtain

$$f(x) \propto x^{1-\alpha}, \quad (3.5.13)$$

or

$$\tilde{L} \propto W^{1-\alpha} \lambda^\alpha \propto W^{\log 4/\log 3}. \quad (3.5.14)$$

Thus, the universal part of the phenomenological result obtained already has been reproduced.

The renormalization group equation is derived from an almost trivial condition, but combining this and the requirement that the $\lambda/\ell$ dependence of the observable can be absorbed in the renormalization constant can produce nontrivial results. The requirement that the divergence can be absorbed in the renormalization coefficient is nothing but the requirement of the existence of a phenomenological description (of the type discussed in Section 3.1). This requirement may not always be fulfilled. However, to require renormalizability is a natural attitude for empirical scientists, when empirically there is a phenomenology. Eventually, the renormalizability must be demonstrated mathematically, but this is a highly nontrivial task, so it is a productive strategy to assume renormalizability and pursue the consequences. The requirement of renormalizability imposes strong constraints on the structure of the observable quantities. Therefore, with other partial information, such as the result of perturbation calculation, we can arrive at fairly nontrivial results. However, the reader must clearly recognize that perturbation is a mere tool and has nothing intrinsically to do with renormalization itself.
To illustrate the preceding paragraph, let us study the von Koch curve from a perturbative point of view. This example can be studied exactly, but such examples are extremely rare, so its perturbative study is not useless. For the actual von Koch curve, when one step of its construction illustrated in Fig. 3.5.1 is performed, the microscopic unit length is changed from $\ell$ to $\ell/3$, and the total length (true length) $L$ to $4L/3$. $4/3$ is fairly different from 1, but let us write this expansion ratio as $e^\epsilon \simeq 1 + \epsilon$ and assume $\epsilon$ to be small. After $n$ construction steps, to the first order in $\epsilon$, we obtain

$$W \rightarrow L = (1 + n\epsilon)W. \quad (3.5.15)$$

This equation is reliable only when $en \ll 1$. In other words, this perturbation cannot be used uniformly in $n$.\(^71\) If $n$ is expressed in terms of $\ell$, we get

$$L = \left\{1 + \epsilon \log_3 \left(\frac{W}{\ell}\right)\right\} W. \quad (3.5.16)$$

Expanding the renormalization coefficient as $Z = 1 + A\epsilon + \cdots$, we choose $A$ to remove the divergence in the $\ell \rightarrow 0$ limit. However, it is convenient to rewrite (3.5.16) as follows, introducing $\lambda$:

$$L = \left\{1 + \epsilon \left[\log_3 \left(\frac{W}{\lambda}\right) + \log_3 \left(\frac{\lambda}{\ell}\right)\right]\right\} W. \quad (3.5.17)$$

Now, the equation corresponding to (3.5.7) can be expanded as

$$\tilde{L} = ZL = \left\{1 + \epsilon \left[A + \log_3 \left(\frac{\lambda}{\ell}\right)\right] + \epsilon \log_3 \left(\frac{W}{\lambda}\right)\right\} W. \quad (3.5.18)$$

Therefore, if we choose $A = -\log_3 (\lambda/\ell)$, the divergence is absorbed into $Z$. The resultant formula

$$\tilde{L} = \left\{1 + \epsilon \log_3 \left(\frac{W}{\lambda}\right)\right\} W \quad (3.5.19)$$

is called the renormalized perturbation series (although here we have only two terms). From this, it is natural to guess

$$\tilde{L} \sim W^{1+\epsilon/\log 3}, \quad (3.5.20)$$

but this exponentiation is required by the renormalization group equation (3.5.11). Only difference from what we did before is that $Z$ is determined only perturbatively here:

$$Z = 1 - \frac{1}{\log 3} \log \frac{\lambda}{\ell}. \quad (3.5.21)$$

Introducing this into the definition of $\alpha$ (3.5.12), we obtain $\alpha = -\epsilon/\log 3$, so (3.5.14) concludes the above ‘natural result’ $\tilde{L} \propto W^{1+\epsilon/\log 3}$. We wish to set $\epsilon = \log 4 - \log 3$; in the present case, the result happens to agree with the exact answer.

\(^71\)In this sense, the term proportional to $n$ corresponds to the secular term in singular perturbation theory (see Section 3.6).
The explanation up to this point simplified the actual situation by assuming as if disparate scales interfere without any restriction and microscopic effects have indefinitely large effects on macroscopic observable. Generally speaking, it is rare that the materials parameter change their values when we change our observation scale. For example, the density and viscosity in the Navier-Stokes equation observed at 1mm and at 1m are the same. If an observable settles down to a particular value in the $\zeta = \lambda/\ell \to \infty$ limit, this means that the effect of the microscopic parameters drop out from macroobservables. However, this contradicts the materialistic diversity of the world. This implies that $\ell \to 0$ and $\lambda \to \infty$ are not the same thing. Therefore, we must conclude that there are more than one dimensionless quantities (at least one more other than $\zeta$) constructed from lengths. Since the properties of materials are determined by microscopic details, between our observation scale $\lambda$ and the atomic scale $\ell$ must be some length $\xi$, and materials constants are determined by the world of the scales up to $\xi$ (they depend on the dimensionless quantity $\xi/\ell$). For example, the correlation length is such a length, which decouples $\lambda$ and $\ell$, so the classical instruction for the dimensional analysis often works.

To extract a macroscopic phenomenology from a microscopic description of a system is, in short, to derive the description of the system at the scale of $\xi$. Then, what determines the correlation length $\xi$? It is not determined solely by microscopic parameters (say, the size of molecules). As can be seen from statistical mechanical calculation, it depends on the energy of the system (temperature) and other fields (say, magnetic field). At the critical point, $\xi$ diverges, so our observation scale directly confronts the microscopic scale ($\xi > \lambda$), but away from criticality, $\xi$ is not at all large.

How much is the idea of RG useful to compute the property of the system at the $\xi$ scale? It is in principle possible to study how observables change when $\xi/\ell$ is varied, but as has been seen in the above simple example, renormalization group idea is really effective, when asymptotic analysis is possible. Whether asymptotic analytical results are useful or not depends on the individual details of the system, because $\xi/\ell$ is not actually very large. There is no general theory that can tell us when asymptotic results become meaningful.

This concludes the explanation of the ABC of renormalization group theory with a bit of general comments. Immediately after this should be detailed illustrations of renormalization procedures for the actual examples mentioned in Section 3.1, but there is no new concept nor new idea needed. Here, some supposedly useful references to learn established methods for critical phenomena and polymer solutions are mentioned in the footnote. The correlation length is small, to suppress the effect of thermal fluctuations, we need a larger scale so that the law of large numbers holds.

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72 If the correlation length is small, to suppress the effect of thermal fluctuations, we need a larger scale so that the law of large numbers holds.

73 (Self-study guide for low-energy-physics renormalization group) To practice the Wilson-Kadanoff approach in critical phenomena the Ising model on a triangular lattice is the best. This is discussed in most statistical mechanics textbooks. An excellent explanation of the St"uckelberg-Petermann approach (the so-called field-theoretical renormalization group theory) for a minimal model called the Ginzburg-Landau model (See Chapter 4) can be found in M. Le Bellac, Quantum and Statistical Field Theory (Oxford 1991). For 2D XY model see Chapter 9 of P. M. Chaikin and T. C. Lubensky, Principles of Condensed Matter Physics (Cambridge UP, 2000). This book also gives a good explanation of renormalization group theory.
accompanying supplementary pages\textsuperscript{74} contain relevant lecture notes with detailed calculated examples.

The rest of this Chapter is devoted to an introductory exposition of renormalized perturbation approach to singular perturbation theory and system reduction.

Appendix 3.5A Introduction to dimensional analysis

Usually, a certain physical quantity, e.g., mass, length, etc., is expressed by numbers indicating how many certain units the quantity corresponds to. The statement, “the length of this stick is 3,” does not make sense; we must say, for example, “The length of this stick is 3 m,” with an appropriate unit. The number itself appearing with a unit has no meaning of its own (3 m and 9.8425\ldots ft are identical). That is, we can freely scale them (by changing the units). However, the statement, “the length ratio of this and that sticks is 4” makes sense irrespective of the adopted unit of the length. The quantity whose numerical expression does not depend on the choice of units is called a dimensionless quantity. The number 4 above is an example. In contrast to the number 3 above which depends not only on the property of the stick alone but also on how we observe it (how we measure it), this 4 does not depend on our observation mode.

An expression of a quantitative relationship among physical observables is a relationship among several numbers. If the relationship exists independent of us (e.g., does not depend on how we look at it), then the truth value of the relationship should at least be free from the choice of units. For example, a relationship that is correct only when the length is measured in meters is not pleasant (it does not capture an essential relationship; at least it is inconvenient).

If we alter the units of a quantity, its numerical value is scaled; the numbers using the same unit must be scaled in the same way (if one does not wish to write units meticulously everywhere, the length must be declared to be measured, say, in inches, and everywhere the convention must be applied). The numerical values corresponding to various quantities measured in different (independent) units may be scaled differently. If the numerical values expressing two quantities always scale identically when the units are changed, we say these two quantities have the same dimension. In other words, scientists and engineers express “independently scalable” as “having distinct dimensions.” Objectively meaningful quantitative relationships should remain true irrespective of the choice of the units. Therefore, such relationships must be expressible in terms of dimensionless quantities (quantities that are not affected by unit changes). To analyze problems according to this requirement is called dimensional analysis.

Let us look at a simple example, the 1D diffusion equation considered in Chapter 1:

\[
\frac{\partial \psi}{\partial t} = D \frac{\partial^2 \psi}{\partial x^2},
\]

(3.5A.1)

where \( t \) is the time, \( x \) is the position coordinate, \( \psi \) is, e.g., the temperature, and \( D \) is the diffusion constant. The integral \( Q(t) \) of \( \psi(x,t) \) over the whole space must be a time-independent constant.

There are three distinct physical units: time, length, and the unit of the quantity \( Q \) in this equation. Let us write these dimensions as \( T \), \( L \), and \( M \), respectively: the usual convention is: \( [X] = \) the dimension of \( X \), so we write \([t] = T\), \([x] = L\), and \([Q] = M\). \( \psi \) is the density of the quantity \( Q \), so in 1-space \([\psi] = M/L\). Differentiation is essentially division, so \([\partial \psi/\partial t] = M/LT\). We wish this equation to hold irrespective of our choice of the units. Then, the both sides of the equation should have the same dimension (we say the equation is dimensionally homogeneous\textsuperscript{75}). Then, \( M/LT = [D]M/L^3 \) is required, so \([D] = L^2/T\).\textsuperscript{76} We can

\textsuperscript{74}http://web.me.com/oono/NonlinearWorldSpringer/Nonlinear_World_Supplements.html

\textsuperscript{75}It is said that the importance of dimensional homogeneity was pointed out first by Fourier.

\textsuperscript{76}The unit of the diffusion constant in SI is indeed m\textsuperscript{2}/s.

assert that objectively meaningful relationships can always be expressed in a homogeneous form.

To perform the dimensional analysis of the diffusion equation (3.5A.1), we must first construct dimensionless quantities (scaling-invariant quantities). Combinations whose powers of $T$, $L$, $M$ are all zero are such quantities: $tD/x^2$ and $\psi x/Q$ or $\psi \sqrt{tD}/Q$.\footnote{There is no other independent combination; other dimensionless quantities are written in terms of the products of some powers of these quantities. The general theory is called Bridgeman’s $\Pi$-theorem, but such a general statement may not be needed; there are three independent dimensions $T$, $L$, $M$ and 5 quantities $Q, \psi, D, t, x$, so $5 - 3 = 2$ independent dimensionless quantities can be constructed. As seen here, the $\Pi$-theorem is a restatement of an elementary theorem in linear algebra.} For example, we can check $[tD/x^2] = T \cdot (L^2/L^2) = 1$. Dimensionless quantities may always be considered as functions of dimensionless quantities (this assertion is easy to understand if we imagine what happens otherwise), so the solution to (3.5A.1) must have the following form:

$$\psi = \frac{Q}{\sqrt{D}} f \left( \frac{x^2}{Dt} \right),$$

where $f$ is a well-behaved function.\footnote{The meaning of “well-behaved” depends on the context, but usually, it means sufficient smoothness (differentiability) and boundedness in the domain under study.} Thus, (3.5A.1) becomes an ordinary differential equation for $f$.

Here, a remark is in order. Whether we may independently scale two different physical quantities or not can be a problem. For example, in the ordinary engineering problems, energy $E$ and mass $m$ have distinct units, but for relativistic phenomena, it is much more natural to regard both to have the same unit. The reason why there is a ‘conversion factor’ $c^2$ in Einstein’s relation $E = mc^2$ is simply due to the non-relativistic convention; the speed of light $c$ is an absolute constant, so $c$ must be regarded as a dimensionless quantity. This concludes that $T$ and $L$ should not be distinguished. In the non-relativistic world, however, this identification is not convenient; we should regard time and length in different categories. (We may perhaps say that $c$ is so large that it cannot be distinguished from infinity, so it drops out of the theory.)

Dimensional analysis is often fundamental. It is clear dimension-analytically that classical physics cannot explain the atomic structure. In a hydrogen atom an electron is bound to a proton by the Coulomb interaction. Therefore, its Newton’s equation of motion reads

$$m \frac{d^2 r}{dt^2} = -\frac{e^2 r}{4\pi \epsilon_0 r^3},$$

where $e$ is the electron charge, and $m$ is the electron mass. $4\pi \epsilon_0$ always appears with $e^2$, so there are only two basic quantities: $m$ and $e^2/4\pi \epsilon_0$. Their dimensions are $[m] = M$, $[e^2/4\pi \epsilon_0] = ML^3/T^2$ (this is required by the dimensional homogeneity of the equation of motion). There is no way to construct a quantity with the unit of length from them. However, if Planck’s constant $h$, whose dimension is $[h] = ML^2/T$ (recall that $h$ times frequency is the photon energy), is relevant as Bohr thought, then, since

$$\frac{m}{e^2/4\pi \epsilon_0} = L^3 T^{-2}, \quad \frac{h}{m} = L^2 T^{-1},$$

we can solve $L$ as $(h/m)^2/(e^2/m\epsilon_0)$:

$$\frac{[h/m]^2/(e^2/m\epsilon_0)}{} = \frac{(L^2 T^{-1})^2/(L^3 T^{-2})}{L}.$$

$\epsilon_0 h^2/m e^2 \times 1/\pi$ is the Bohr radius $= 0.53$ Å $= 0.053$ nm (nanometer; $1$ nm $= 10^{-9}$ m), which is roughly the size of the hydrogen atom in its ground state. Actually, this argument convinced Bohr that $h$ is necessary.

Discussion 3.5A.1 Derive Kepler’s Third Law dimensional analytically. □

Discussion 3.5A.2 An interesting book by Migdal\footnote{A. B. Migdal, Qualitative methods in quantum theory (translated by A J. Leggett) (Westview Press, 2000).} begins with a dimensional analytical ‘proof’ of Pythagoras’ theorem. The argument relies on the expression of the area $S = a^2 f(\alpha)$ of an orthogonal triangle whose smallest angle is $\alpha$ with the length of its hypotenuse being $a$ (Fig. 3.5A.1). As can be seen from the figure
\[ a^2 + b^2 = c^2. \] (Can we prove \( S = a^2 f(\alpha) \) within the Euclidean geometry?\(^{780}\) □

\[ a + b = c. \]

Figure 3.5.1: Pythagoras’ theorem. This dimensional analytic result does not contradict the result of Euclidean geometry. However, is this a respectable proof of the theorem? How can you demonstrate \( S = a^2 f(\alpha) \)? Is Migdal logical?

Appendix 3.5.B Dimensional analysis and renormalization

Any relationship meaningful in physics may be written as a relationship among dimensionless quantities (the quantities indifferent of the choice of units, see Appendix 3.5A). Such relationships generally have the following form:

\[ \Pi = f(\Pi_0, \Pi_1, \ldots, \Pi_n), \tag{3.5B.1} \]

where \( \Pi \) and \( \Pi_i \) \( (i = 0, 1, \ldots, n) \) are dimensionless quantities. In the following, we review in a more general fashion what we have seen in this section. If we read a standard explanation of dimensional analysis, it is usually stated that among the dimensionless quantities “extremely large ones and ones too close to zero may be ignored.” This recommendation may be formalized as follows. Suppose \( \Pi_0 \) is very small. “The following limit

\[ f(0, \Pi_1, \ldots, \Pi_n) = \lim_{\Pi_0 \to 0} f(\Pi_0, \Pi_1, \ldots, \Pi_n) \tag{3.5B.2} \]

‘exists,’ so we may consider \( \Pi \) without \( \Pi_0 \).” This procedure is the assertion that we may ignore any interference between disparate scales.

The key effect of nonlinearity arises when this assertion is false. If the limit in the right-hand side of (3.5B.2) does not exist, essentially nonlinear phenomena can be observed. Let us assume that all the dimensionless quantities in (3.5B.1) include a quantity that cannot be observed directly (quantities that cannot be operationally defined; the unit segment length \( \ell \) of the von Koch curve or the monomer unit of polymers are examples). In the \( \Pi_0 \to 0 \) limit, these dimensionless quantities may all diverge (going to zero is also a kind of divergence in the present context). Let us separate \( \Pi_0 \) into the part dependent on the observation condition \( \Pi_O(\lambda) \) and the part that goes to zero \( \Pi_{NO}(\lambda) \) (recall the separation performed in (3.5.17)), where \( \lambda \) is the parameter specifying the scale of observation. If we describe a phenomenon that can be directly observed, its phenomenology must be written in terms of renormalized quantities. Let us introduce renormalized quantities denoted by the suffix \( R \) and the renormalization constant \( Z_i \) as follows:

\[ \Pi_R = Z \Pi, \quad \Pi_{Ri} = Z_i \Pi_i. \tag{3.5B.3} \]

The renormalization constants are chosen to absorb divergences in the \( \Pi_{NO} \to 0 \) limit. Introducing the above definitions into (3.5B.2), we may convert the relation to the relation among renormalized quantities. Suppose the result reads:

\[ \Pi_R = F(\Pi_{R1}, \ldots, \Pi_{Rn}). \tag{3.5B.4} \]

\(^{780}\)The argument here implies only that the key issue of the proof is, logically speaking, to demonstrate \( S = a^2 f(\alpha) \). This requires far more subtle theories than the usual elementary proof(s) of the theorem (we need at least Book 5 (ratios and proportions) and the next volume about similar figures).
The renormalization group equation comes from the fact that \( \lambda \) has nothing to do with the system being observed,

\[
\lambda \frac{\partial \Pi}{\partial \lambda} = 0.
\] (3.5B.5)

Rewriting (3.5B.4) as

\[
\Pi = Z^{-1} F(Z_1 \Pi_1, \cdots, Z_n \Pi_n)
\] (3.5B.6)

and introducing this into the renormalization group equation (3.5B.5), we obtain with the aid of the chain rule

\[-\alpha \Pi + \sum_i \alpha_i \Pi_i \frac{\partial \Pi}{\partial \Pi_i} = 0.\] (3.5B.7)

Here,

\[
\alpha \equiv \frac{\partial \log Z}{\partial \log \lambda}, \quad \alpha_i \equiv \frac{\partial \log Z_i}{\partial \log \lambda}.
\] (3.5B.8)

In the \( \Pi_{NO} \to 0 \) limit, let us assume that \( \alpha, \alpha_i \) have limits.

The renormalization group equation (3.5B.7) may be solved with the aid of the method of characteristics (see Note 3.5B.1 at the end of this appendix). The characteristic equation becomes

\[
\frac{d\Pi_i}{d\rho} = \frac{\alpha_i \Pi_i}{\rho}, \\
\frac{d\Pi}{d\rho} = \frac{\alpha \Pi}{\rho},
\] (3.5B.9)

with the introduction of the dilation parameter \( \rho \), which is introduced to parameterize the characteristic curve. To solve this, we first separate it as

\[
\frac{d\Pi_i}{d\rho} = \alpha_i \frac{\Pi_i}{\rho}, \\
\frac{d\Pi}{d\rho} = \alpha \frac{\Pi}{\rho},
\] (3.5B.10)

and then solve each as follows:

\[
\Pi_i = C_i \rho^{\alpha_i}, \quad \Pi = C \rho^\alpha.
\] (3.5B.12)

Using the fact that the general relation among the integral constants \( C, C_i \) becomes the general solution, we obtain the general solution to (3.5B.9) as

\[
\Pi = \rho^\alpha \Phi((\Pi_{R1} / \Pi_{R1})^\alpha_i, \cdots, (\Pi_{Rn} / \Pi_{R1})^\alpha_n),
\] (3.5B.13)

where \( \Phi \) is a well-behaved function. This relation must hold for any (positive) \( \rho \). Therefore, we may set, for example, \( \rho = \Pi_{R1}^{1/\alpha_1} \). This gives the following relation:

\[
\Pi = \Pi_{R1}^{\alpha_i/\alpha_1} g(\Pi_{R2}^\alpha / \Pi_{R1}^\alpha, \cdots, \Pi_{Rn}^\alpha / \Pi_{R1}^\alpha).
\] (3.5B.14)

Here, \( g \) is also a well-behaved function and exponents \( \alpha, \alpha_i \) cannot be determined by dimensional analysis (since all the quantities are already dimensionless).

Even if the limit (3.5B.2) does not exist, Barenblatt\(^{81}\) realized that still some sort of limit can be taken, and demonstrated that solving problems assuming the form of solution as (3.5B.14) is often successful.\(^{82}\) He called this limit the intermediate asymptotics (of the second kind). The key ingredient of this concept is the relation that holds for a long time before the system reaches its really long time asymptotic state, which is, for many systems, a sort of dead state. For example, if we quench a binary alloy from its high temperature well-mixed state to a low temperature state where phase separation occurs (see Example 3.1.4 and the next Chapter), after a long time the phase separation is complete and no change would occur, but during the intermediate time scale before reaching this really asymptotic quiescent state, locally segregated

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81 A brother of Sinai, who appeared in Chapter 2.
phases coarsen with its size proportional to the time to the $1/3$ power. Such a law is the intermediate asymptotic law. In this example, by increasing the system size, we can indefinitely widen the intermediate time span where this law holds, so this intermediate limit is even practically a very meaningful limit. In the actual calculation based on the intermediate asymptotic limit theory the form (3.5B.14) is assumed and the exponents are selected (as a nonlinear eigenvalue problem) so that there is a solution. In any case, this theory does not ask the question why the form (3.5B.14) is natural; it is an assumption.

It may not be necessary any more, but let us review the von Koch curve. As before, let us write the actual length of the curve as $L$, the width as $W$, and the length of the smallest segment as $\ell$. There are two dimensionless quantities $L/W$ and $\ell/W$, so dimensional analysis tells us that

$$L/W = f(\ell/W). \tag{3.5B.15}$$

The right-hand side diverges in the $\ell \to 0$ limit, so the equation corresponding to (3.5B.14) is

$$L/W \propto (\ell/W)^{-\alpha/\alpha_1}. \tag{3.5B.16}$$

That is, we obtained $L \propto W^{1+\alpha/\alpha_1}$ again, naturally. However, we cannot determine the exponents within this formalism.

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Note 3.5B.1 Method of characteristic equation
The idea of the method is used to solve the general first-order partial differential equations, but here we confine ourselves to quasilinear first order partial differential equations on a domain $U$ in an $n$-dimensional space:

$$\sum_{i=1}^n b_i(u, x) \frac{\partial u}{\partial x_i} = c(u, x). \tag{3.5B.17}$$

83  "The history of renormalization group theory for differential equations" The author learned the problem of intermediate asymptotics from his long time collaborator Nigel Goldenfeld in conjunction to the Barenblatt equation:

$$\frac{\partial u}{\partial t} = \frac{1}{2} \left[ 1 + \epsilon \Theta \left( \frac{\partial u}{\partial t} \right) \right] \frac{\partial^2 u}{\partial x^2},$$

where $\Theta$ is Heaviside’s step function and $\epsilon$ is a positive constant. He had been interested in the intermediate asymptotics for some years, but since the author, not knowing the theory of the intermediate asymptotics at all, did not pay much attention. However, on one day in 1988 or 89, his student gave a seminar on the Barenblatt equation, and the author knew the intermediate asymptotics problem. After a couple of days of hesitation (in this example, the perturbation term is so singular that one needs desperation to summon courage to use perturbation) the author started perturbation calculation and immediately the structure of the problem became clear. This was only possible by the training the author got from another long time collaborator Takao Ohta while studying polymer solutions (see Example 3.1.3). N. Goldenfeld, Lectures on phase transitions and renormalization group (Addison Wesley, 1992) Chapter 10 explains the above result [the original paper is N. D. Goldenfeld, O. Martin and Y. Oono, “Intermediate asymptotics and renormalization group theory,” J. Scientific Comp. 4, 355 (1989)].

The next important progress came when the author realized that the Burgers equation was a renormalization group equation [L.-Y. Chen, N. Goldenfeld, Y. Oono, and G. Paquette, “Selection, stability and renormalization,” Physica A 204, 111 (1993)]. This told him that singular perturbation theory and reductive perturbation theory could be unified within the renormalization group theory. See Section 3.7 and its Appendix for an outline). The reductive perturbation was familiar to the author because he was taught theoretical physics by Tomoji Yamada when he was at Kyushu University, where I could observe his collaborator Yoshiki Kuramoto from distance as well. After all, the relation to reductive perturbation was the essence of the renormalization group theory of differential equations. This view was perfectly vindicated by the next breakthrough, Hayato Chiba’s qualitative theory of renormalization group equations [H. Chiba, “$C^1$ approximation of vector fields based on the renormalization group method,” SIAM J. Appl. Dym. Syst. 7, 895 (2008)].
A solution to this equation \( u = f(x) \) describes a surface floating on \( U \) (Fig. 3.5B.1; it is a hypersurface in the \((n + 1)\)-space). If we draw a curve \( x = x(s) \) in \( U \) parameterized by \( s \), a curve \( z(s) = u(x(s)) \) is drawn on the solution surface (see Fig. 3.5B.1; the curve is actually given by the position vector \((x(s), z(s))\)). Conversely, if this curve \( z \) is known, we can construct a solution surface.

![Figure 3.5B.1](image)

**Figure 3.5B.1:** The gray curved surface is a solution surface. The curve \( x \) on the plane is \( x = x(s) \), and the curve above it in the solution surface is \( z(s) = u(x(s)) \). [Actually, the trajectory of the vector \((x(s), z(s))\).]

Since

\[
\frac{du}{ds} = \sum_i \frac{d x_i}{ds} \frac{\partial u}{\partial x_i},
\]  

(3.5B.18)

comparing this with (3.5B.17) and choosing a curve \( x \) satisfying

\[
\frac{d x_i}{ds} = b_i(u, x),
\]  

(3.5B.19)

we can write (3.5B.18) as

\[
\frac{du}{ds} = c(u, x).
\]  

(3.5B.20)

Thus obtained differential equations (3.5B.19) and (3.5B.20) are called characteristic equations. We can write them as

\[
\frac{d x_1}{b_1} = \cdots = \frac{d x_n}{b_n} = \frac{d u}{c} = \left( \frac{d \rho}{\rho} \right),
\]  

(3.5B.21)

where \( \rho = e^s \) is the dilation parameter. The differential equation (3.5B.9) is an example. From this we obtain \( n \) equations. Integrating them gives the solution: \( F_1(x, u) = C_1, \ldots, F_n(x, u) = C_n \), containing \( n \) integration constants. Except for special cases, these equations describe \( n \) hypersurfaces mutually in the general position, so the common set of all these surfaces determines a curve floating on \( U \). It is parameterized by \( C_1, \ldots, C_n \). Since this curve must be embedded in a solution surface, moving this curve should make a solution surface. However, if we vary all the parameters \( C_1, \ldots, C_n \) independently, the curve would fill a part of the \((n + 1)\)-space. A hypersurface is a subset of codimension 1, so if we specify one relationship among \( C_1, \ldots, C_n \), then the curves determined by \( C_1, \ldots, C_n \) should as a whole determine a hypersurface. Therefore, choosing a well-behaved function \( G \), we demand

\[
G(C_1, \ldots, C_n) = 0.
\]  

(3.5B.22)

In other words,

\[
G(F_1(x, u), \ldots, F_n(x, u)) = 0
\]  

(3.5B.23)

must be the general expression of the hypersurface determined by (3.5B.17).