Kadanoff’s explanation of scaling

0.0.1 Kadanoff construction
Without any simulation Kadanoff (1937-2015) completely understood the structure exhibited in Ashton’s video and succeeded in elucidating the general features of critical phenomena with an ingenious intuitive picture Fig. 0.0.1.

Figure 0.0.1: A: The Kadanoff construction. ‘Shrinking’ corresponds to looking at the system from distance with fixed eyesight, that is, scaling + coarse-graining. The outcome corresponds to the system away from the critical point: the correlation length $\xi$ becomes smaller. B: If we step back and the distance between us and the sample becomes $\ell$ times as large as the original distance (in the figure $\ell = 2$), the actual linear dimension of the minimum discernible volume becomes $\ell$-times as large as the original minimum discernible volume.

If the original system has a temperature $\tau = (T - T_c)/T_c$ and magnetic field $h$, then from our stepped-back point of view the system looks as if it has these parameters scaled (increased; farther away from the critical point) to $\tau \ell^{y_1}$ and $h \ell^{y_2}$; the exponents $y_1$ and $y_2$ must be positive, where $\ell$ is the shrinking rate ($> 1$). This is a guess or hypothesis, but seems to explain everything neatly as we will see below.

0.0.2 Scaling law
Let us write $m = M(\tau, h)$ (this is the equation of state for the magnetic system). After one stepping-back, the volume of the region recognized as a unit cube to us would be actually the cube with edge $\ell$ (see Fig 0.0.1 Right) before stepping back.

Let us put $'$ to the quantities observed after stepping back. We look at the magnetic energy stored in the minimum discernible block $h'm'$ (after shrinking). The energy should be a much better additive quantity than the local magnetic moment (since energy is additive even microscopically), so we expect

$$h'm' = \ell^d hm.$$  (0.0.1)
Since \( h' = h\ell^{y_2} \), we obtain

\[ m = \ell^{-d} (h'/h)m' = \ell^{y_2-d} M(\tau', h') = \ell^{y_2-d} M(\tau\ell^{y_1}, h\ell^{y_2}). \]  

This is the scaling relation for the equation of state. It should be clearly recognized that this is an identity that holds for any positive number \( \ell \). Therefore, we may set \( |\tau|\ell^{y_1} = 1 \). Thus, we obtain from (0.0.2) (\( \tau < 0 \) to have non-zero magnetization)

\[ m(\tau, 0) = |\tau|^{(d-y_2)/y_1} m(-1, 0). \]  

That is,

\[ \beta = \frac{d-y_2}{y_1}. \]  

We can also conclude from the derivative of (0.0.2) with respect to \( h \):

\[ \gamma = \frac{2y_2-d}{y_1}. \]

1.3 Critical exponent equality

To obtain \( \alpha \) we must compute the specific heat, which is available as the second derivative of the free energy with respect to \( T \) (recall ??). The (singular part of the) free energy \( f_s = F_s(\tau, h) \) per minimum discernible volume unit scales as

\[ f_s = F_s(\tau, h) = \ell^{-d} F_s(\tau\ell^{y_1}, h\ell^{y_2}). \]

This comes from \( f_s' = \ell^d f_s \) due to the extensivity of the free energy. If we differentiate (0.0.6) with \( h \), we get (0.0.2). Differentiating (0.0.6) twice with respect to \( \tau \) (that is, \( T \)), we obtain

\[ C(\tau, h) = \ell^{2y_1-d} C(\tau\ell^{y_1}, h\ell^{y_2}). \]

Therefore,

\[ \alpha = \frac{2y_1-d}{y_1}. \]

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2The free energy itself has a large nonsingular part that does not contribute to the singular behaviors near the critical point (Discussion 13.4).
From (0.0.4), (0.0.5) and (0.0.8) we obtain Rushbrooke’s equality:

\[ \alpha + 2\beta + \gamma = 2. \]  

(0.0.9)

0.0.4 Renormalization group transformation

Kadanoff’s idea (Kadanoff construction) consists of two parts: coarse-graining and scaling (shrinking). The crux of the idea is: if the system is at the critical point, the configuration is invariant under coarse-graining \( \mathcal{K} \) with an appropriate scaling \( \mathcal{S} \). That is, if we define \( \mathcal{R} = \mathcal{K}\mathcal{S} \), then thermodynamic observables (densities and fields) are invariant under the application of \( \mathcal{R} \) at \( T_c \). To apply \( \mathcal{R} \) is to observe the system from distance with a fixed eyesight. Fig. 0.0.2 Left illustrates how iterative operations of \( \mathcal{R} \) drive the statistical configurations at various temperatures.

Operating \( \mathcal{R} \) is called a renormalization group transformation. We can understand its iterative applications as multiplication of \( \mathcal{R} \); doing nothing corresponds to the unit element. Therefore, the totality of the renormalization group transformations is informally called a renormalization group.\(^3\) According to Kadanoff’s original idea, the image due to \( \mathcal{R} \) is the same system under a different condition (e.g., at a different temperature), so we may understand that \( \mathcal{R} \) transforms a thermodynamic state into another (of the same system); then, we may imagine that successive applications of \( \mathcal{R} \) define a flow on the phase diagram of the same materials system under study. This view is illustrated in Fig. 0.0.2 Right.\(^4\)

0.0.5 Renormalization group fixed point

At the fixed point \( \mathcal{R}\xi = \xi \) should hold for the correlation length \( \xi \). Since \( \mathcal{S} \) definitely shrinks the system, this condition is satisfied only if \( \xi = 0 \) or \( \xi = \infty \). That is, the phases without spatial correlation at all or critical points are the only possible fixed points. Notice that if we understand these fixed points, we understand the general

\(^3\)The inverse may not be defined, so it is usually a monoid. For the concept of ‘group’ see Section 28

\(^4\)As we will soon see, this flow does not generally flow on the phase diagram (of a given material). In terms of Fig. 0.0.2 Left, the flow diagram exhibits what happens to the ‘actual’ configurations. The renormalization flows move as \( n = 1, 2, \cdots \) to the left, starting from the ‘actual slice’; note, however, the obtained configurations are generally not exactly realized by any state in the phase diagram of the system under study. The flows in Fig. 0.0.2 Right are, intuitively, an approximate projection of these RG flow lines onto the actual system.
Figure 0.0.2: **Left:** The result of Kadanoff construction, or the real space renormalization group transformation. For simplicity, \( h = 0 \) (i.e., on the right phase diagrams we study the system only along the curve BC in Upper Right). Here, \( \tau = (T - T_c)/T_c \) and \( n \) is the number of times we operate the renormalization group transformation \( \mathcal{R} \); we start from the actual configurations \( (n = 0) \) at various temperatures. As \( \mathcal{R} \) is applied successively, the configurations are transformed as the arrows indicate. The leftmost vertical line denotes the destination after many applications of \( \mathcal{R} \). a, b, c correspond to the trajectories a, b, c in Fig. 0.0.3. Only when the starting point is just right, the system can stay at \( \tau \sim 0 \). The low temperature states are driven to one of the ordered phases at \( T = 0 \); in the illustration it happens to be totally ‘down.’ If the starting point is \( T > T_c \), the state is driven to \( T = \infty \) state. **Right:** RG flows ‘projected’ (see the text) on the phase diagram of the (Ising) magnet. There are five ultimate destinations (high temperature limit, phase boundary, critical point, all up and all down low temperature states).

structure of the phase diagram. The ordinary bulk phases from our macroscopic point of view do not have any appreciable correlation distance, so they are close to the \( \xi = 0 \) fixed points. To understand their macroscopic properties we need not worry (qualitatively) about spatial correlations of fluctuations (see footnote 9). This is the reason why the so-called mean-field theory (see Section 26) is useful. Thus, to understand the phase diagram, we use mean field theory to understand the bulk phases not too close to the critical points,\(^5\) and use renormalization group theory to

\(^5\)This does not mean that we can use the original microscopic Hamiltonian when we utilize a mean-field approach; we must use an appropriately renormalized Hamiltonian. Therefore, a precise statement is: there is a (model) Hamiltonian (with short-range interactions) that can be used to
understand the features near the critical points.

0.0.6 Renormalization group flow (RG flow)
We may interpret the renormalization group transformation as a map from a (generalized) canonical distribution $\mu$ to another (generalized) canonical distribution $\mu' = \mathcal{R}\mu$. We can imagine effective Hamiltonians $H$ and $H'$ (it is customary that $\beta$ is absorbed in $H's$) according to

$$
\mu = \frac{1}{Z} e^{-H}, \quad \mu' = \frac{1}{Z'} e^{-H'}.
$$

(0.0.10)

We may write $H' = \mathcal{R}H$. Therefore, we can imagine that successive applications of $\mathcal{R}$ defines a flow (RG flow) in the space of Hamiltonians (or models or systems). This idea is illustrated in Fig. 0.0.3 (Fig. 0.0.2 Left actually illustrates the pattern changes along $a$, $b$ or $c$ in Fig. 0.0.3). In Fig. 0.0.3 $H^*$ is a fixed point with an infinite correlation length of the RG flow. Its stable manifold\(^6\) is called the critical surface. The Hamiltonian of the actual material, say, magnet A, changes (do not forget that $\beta$ is included in the definition of the Hamiltonian in (0.0.10)) as the temperature changes along the trajectory denoted by the curve with ‘magnet A.’ It crosses the critical surface at its critical temperature. The renormalization transformation uses the actual microscopic Hamiltonian of magnet A at various temperatures as its initial conditions. Three representative RG flows for magnet A are depicted. ‘a’ is slightly above the critical temperature, ‘b’ exactly at $T_c$ of magnet A (‘b’ is the corresponding RG trajectory for magnet B, a different material; both b and b’ are on the critical surface), ‘c’ slightly below the critical temperature (these $a$, $b$, $c$ correspond to those in Fig. 0.0.2 Left). Do not confuse the trajectory (black curve) of the actual microscopic system as temperature changes and the trajectories (successive arrows; RG flow) produced by the RG transformation.

If we understand $H^*$, we understand all the universal features of the critical behaviors of all the magnets crossing its critical surface.

\(^6\)\textit{(Stable manifold)} For a fixed point $x$, the totality of points $y$ flowing into $x$ is called the \textit{stable manifold} of $x$. 

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\(^5\)\textit{describe the macroscopic features of a bulk phase with the aid of a mean-field approach.}
0.0.7 Central limit theorem and renormalization\textsuperscript{7}
We have realized that near the second order phase transition/critical point, fluctuations become large, and also strongly correlated. How can we handle such a strongly correlated fluctuating system? We wish to know the distribution of fluctuations at the mesoscopic scale (because the correlation lengths are at the mesoscopic scale). If we understand the mesoscopic fluctuation statistics, we should be able to compute macroscopic observables. How can we study the mesoscopic fluctuation statistics? The usual central limit theorem is the tool to study mesoscopic fluctuations if random variables are iid (= independently and identically distributed):

If $X_i$ are assumed to be iid with a finite variance $V$ and zero mean, the density distribution function of $S_N/\sqrt{N}$ converges to a Gaussian distribution function with mean 0 and variance $V$.

According to the Kadanoff construction, we make a block spin (by summing nearby spins) and then scale it appropriately as we see in (0.0.2). Blocking corresponds

to constructing the sum $S_N$, and scaling corresponds to dividing with $\sqrt{N}$. At the renormalization group fixed point we expect the ‘blocked and scaled’ spins obey a definite distribution function. Thus, we understand that renormalization group procedure is an extension of the ordinary central limit theorem to highly correlated random variables. Unfortunately, mathematically, this generalization is still in its infancy.

Let us have a taste of quantitative realization of the renormalization idea.

0.0.8 Detailed illustration of real space renormalization group calculation

Kadanoff’s idea, which may be summarized as follows, allows us to compute, e.g., critical exponents: Introduce some method $\mathcal{K}$ to coarse-grain the system. This method also dictates the spatial scale reduction rate $\ell$. The coarse-graining method $\mathcal{K}$ may be understood as a map from a configuration $S$ (this may be a field or spin configuration $\{s_i\}$) of the original system to a configuration of the reduced system. Fig. 0.0.4 illustrates two examples. The important point of $\mathcal{K}$ is that it is a map: given a configuration $S$, $\mathcal{K}(S)$ is unique. However, it is not an injection (one-to-one map), since it is a kind of coarse-graining.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig004.png}
\caption{Left: Decimation of 1-Ising model, $\ell = 2$ (see Q33.1); Right: Blocking of 3 spins of the triangular lattice 2-Ising model, $\ell = \sqrt{3}$. The value of the block spin is determined by the majority rule: the block spin is up (down) if two or more spins being blocked are up (down).}
\end{figure}

Triangular lattice 2-Ising coarse-graining:

Let us study the triangular lattice Ising model. It is generally the case that coarse-

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8To understand renormalization group approaches the best way is to follow a few examples to nurture the reader’s intuition; Leo Kadanoff told the author that it was easy to invent an RG if we knew the answer (= the system behavior). Chaikin, P. M. and Lubensky, T. C., (1995). *Principles of condensed matter physics*, Cambridge, Cambridge University Press contains excellent explanations and examples.
graining produces multi-spin interactions, even if the original model contains only binary spin interactions as in the present example. However, we wish to be as simple as possible, so we use a (crude but still interesting) approximation that under $K$ illustrated in Fig. 0.0.4 Right, the Hamiltonian preserves its shape (that is, we assume that the RG flow does not leave the phase diagram of this particular system; Recall Fig. 0.0.2):\footnote{More accurate handling of this problem can be seen in Niemeijer, Th. and van Leeuwen, J. M. J. (1973). Wilson theory for spin systems on a triangular lattice, \textit{Phys. Rev. Lett.}, \textbf{31}, 1411-1414.}

$$H = \sum Ks_is_j + hs_i \rightarrow H' = \sum K's'_\alpha s'_\beta + h's'_\alpha,$$  \hspace{1cm} (0.0.11)

where $s'_\alpha$, etc. denote the block spins defined by the majority rule: if two or more spins are up (down) in the block, the block spin is up (down).

**How to specify block spins:**

Fig. 0.0.5 explains the block spins more explicitly. For simplicity, let us study the small $h$ case; we ignore its effect on the coarse-grained coupling constant. Since we are interested in the macroscopic global behavior of the mode, we need not worry about the intrablock spin interactions.\footnote{They shift the origin of the free energy, but it has nothing to do with the correlation length, so they correspond to the non-singular part of the free energy. Recall that we discussed the singular part of the free energy; we are picking up the singular part only. See \textit{0.0.3}.} Therefore, the ‘block spin $\alpha$’-'block spin $\beta$' interaction energy must be equal to the sum of the interaction energies among the original spins belonging to different blocks. As can be seen from Fig. 0.0.5, we may demand

$$K's'_\alpha s'_\beta = K(s_{\alpha_2} s_{\beta_1} + s_{\alpha_3} s_{\beta_1})$$  \hspace{1cm} (0.0.12)

on the average (we cannot demand this exactly). That is, the block spin $\alpha$-$\beta$ interaction is supported by two ‘actual’ interactions: interactions between $\beta_1$ spin and $\alpha_2$ and $\alpha_3$ spins.

**Spin-block spin relation:**

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure0.0.5.png}
\caption{Triangular lattice and the block spins $\alpha$ and $\beta$. 1, 2, 3 denote the original spins (small black dots). The rounded triangles denote block spins, and small gray disks indicate the positions of the block spins.}
\end{figure}
If we wish to relate $K$ and $K'$, we must relate $s$ and $s'$. The basic idea is that near the critical point the correlation length $\xi$ is large, so

$$K' s'_\alpha s'_\beta = K(\langle s_{a2} \rangle s'_\alpha \langle s_{\beta 1} \rangle s'_\beta + \langle s_{a3} \rangle s'_\alpha \langle s_{\beta 1} \rangle s'_\beta),$$

(0.0.13)

where $\langle s \rangle_{s'}$ is the average of the original spin $s$ in the block spin whose value is $s'$ (a conditional average), and

$$s'_\alpha = \text{sgn}(\langle s_{\alpha 1} \rangle s'_\alpha).$$

(0.0.14)

The following table tells us the original spin configuration compatible with $s'_\alpha = +1$ (i.e., the majority up; $s_{\alpha 1}$ spin is circled). The last line in the table is the intra-block energy of the block spin that determines how a particular internal configuration is likely. Therefore, we obtain

$$\langle s_{\alpha 1} \rangle s'_\alpha = \phi(K) s'_\alpha.$$  

(0.0.16)

**$K \to K'$ relation:**  
(0.0.13) now reads

$$K' s'_\alpha s'_\beta = 2K \phi(K)^2 s'_\alpha s'_\beta,$$

(0.0.17)

or

$$K' = 2K \phi(K)^2.$$  

(0.0.18)

**$h \to h'$ relation:**  
Since we have assumed that $h$ is small, we may simply ignore its effect on $K'$, and we require

$$h' s'_\alpha = h(s_{\alpha 1} + s_{a2} + s_{a3}),$$

(0.0.19)

so we immediately obtain

$$h' = 3h \phi(K).$$

(0.0.20)

$\mathcal{R}$ has been constructed:
This completes our construction of $\mathcal{R} : (K, h) \to (K', h')$ with $\ell = \sqrt{3}$ (from the geometry: Fig. 0.0.5).

**Fixed points of $\mathcal{R}$:**
Let us look for fixed points of $\mathcal{R}, (K_F, h_F)$, determined by

$$K_F = 2K_F \phi(K_F)^2, \quad h_F = 3h_F \phi(K_F).$$ (0.0.21)

$K_F = 0$ is certainly a solution, but $\phi = 1/\sqrt{2}$ gives $K^* = (1/4) \log(1 + 2\sqrt{2}) \approx 0.3356 \cdots$. For all $K_F h_F = 0$ is a solution. There is no other finite solution.\(^{11}\) That is, $(K, h) = (0, 0)$ or $(K^*, 0)$ is the fixed point.\(^{12}\) From the correspondence explained in Fig. 0.0.6 the unstable fixed point $(K^*, 0)$ in the $(K, h)$-plane corresponds to the critical point seen from far away; The thick $K$ axis corresponds to the thick curve through $H^*$ in Fig. 0.0.3. **Flow near the fixed point; linear approximation to**

![Figure 0.0.6: The RG flow for the triangular lattice Ising model in the $(K, h)$-plane. The black dot denotes the location of the nontrivial fixed point $(K^*, 0)$. The origin is also a fixed point. This figure corresponds to Fig. 0.0.2 Right; larger $K$ corresponds to lower temperature. The black dot corresponds to $H^*$ and the $K$ axis to the thick curve through $H^*$ in Fig. 0.0.3.](image)

**$\mathcal{R}$:**
Although we studied both $\tau$ and $h$, let us study $h = 0$ as before and the flow along the unstable manifold of $H^*$ (the thick curve in Fig. 0.0.3). As can be seen from $\mathcal{R} \tau = \tau \ell^{y_1}$ and since $K - K^*$ is essentially $\tau$, we must study the local behavior of $\mathcal{R}$ near the critical fixed point. Let $K = K^* + \delta K$ and $K' = K^* + \delta K'$. Then the linear approximation of $\mathcal{R}$ along the $K$ curve is

$$\mathcal{R}(K^* + \delta K) = K^* + \left( \frac{d\mathcal{R}}{dK} \right)_{at \ K^*} \delta K = K^* + \delta K'.$$ (0.0.22)

We may identify ($K'$ is given by (0.0.18))

$$\ell^{y_1} = \left. \frac{dK'}{dK} \right|_{K=K^*} = 1.634 \cdots.$$ (0.0.23)

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\(^{11}\)Don’t divide the equation with zero.

\(^{12}\) $K_F = \infty$ is a fixed point corresponding to the ordered phases or $T = 0$. 
Therefore, since \( \ell = \sqrt{3} \),

\[
y_1 = \log \frac{1.634}{\log \sqrt{3}} \simeq 0.8939 \ldots
\]

Its exact value is 1 (related to the critical exponent \( \alpha \)).\(^{13}\) The reader may think the result is not impressive (the mean field theory gives 2).

\[\text{0.0.9 Universality, trivial and nontrivial}\]

The reader might say we already know examples of universality. For example, we know \( PV = 2E/3 \) for any ideal gas irrespective of statistics, if the spatial dimensionality is 3, and the dispersion relation is \( \varepsilon \propto p^2 \) (Section 28). Or we know \( PV = E/3 \) for phonons and photons in 3-space. This is quite universal. However, it is due to the universality (common quantitative feature) of the elementary entities making up the systems. In this sense, universality is unsurprising, and trivial.

In contradistinction, the universality near the critical point is obviously not due to some common quantitative features at the microscopic level. Of course, the system must exhibit a critical phenomenon, but we use only three features: the interaction is short-ranged, the order parameter is a scalar, and the system is 3D. Thus, the reason for the universality is not in the common nature of the system constituents.

Furthermore, the response to system changes is quite different from the trivial case above. If one adds certain interactions, the ‘trivial’ universality is lost in infinitely different ways according to the infinitely different perturbations. In contrast, in the case of the critical phenomena, if one turns on perturbations modifying interactions, \( T_c \) changes sensitively and also the actual values of susceptibilities are altered, but the main features (e.g., critical exponents) do not change. Thus, the universality of the second-order phase transition deserves to be called the \textit{genuine universality}.

\[\text{0.0.10 What is statistical mechanics for?}\]

The phase transition points (e.g., \( T_c \)) and the values of susceptibilities sensitively depend on materialistic details as mentioned above. We also noted that there is no use of theory to study chemical equilibrium constants in \( \textnormal{??} \). Generally speaking, it is impossible to calculate materials constants very accurately through implementing the theoretical formalism of statistical thermodynamics. Then, what is statistical mechanics for? Statistical mechanics should try to understand (and compute) universal

\(^{13}\)Actually, \( 1/y_1 = \nu \), the exponent for the correlation length.
features of many-body systems that are insensitive to quantitative details. Needless to say, demonstrating the existence of some features (say, a phase transition) is an important target of statistical mechanics. As we will see soon in the case of critical phenomena there is a hope that statistical mechanics can obtain universal features quantitatively. Actually, it is fair to say that the true role of statistical mechanics was consciously recognized as the study of universality and not of fetish details (as the actual value of $T_c$) through the study of critical phenomena.