1. [Elementary combinatorics questions]

Although combinatorics is a side issue of probability theory, if all the elementary events are equally probable, counting becomes the main problem of evaluation of probabilities, so let us look at its rudiments. You must have read Appendix 3A up to the multinomial theorem.

(1) How many ways
(i) to put 3 distinguishable balls into three distinguishable boxes? More generally, how about $n$ balls and $M$ boxes?
(ii) to put 3 indistinguishable balls into three distinguishable boxes? General case?
   This is to distribute energy quanta to three different molecules.
(iii) to put 3 distinguishable balls into three indistinguishable boxes? General case?
   This is a grouping problem.\(^1\)
(iv) to put 3 indistinguishable balls into three indistinguishable boxes?
   This is related to a partition question of integers.\(^2\)

(2) All the elementary particles of the same kind [and all the molecules of the same chemical species with the same internal state] are indistinguishable.

If the number density is extremely low, then whether these particles are distinguishable or not is virtually irrelevant, so we may handle them just as ‘marbles.’

However, the number density is not too small, their indistinguishability manifests itself. For example, suppose you have two particles and two distinguishable boxes (= one-particle states). There is only one way to put one particle each in each box (not two as the case of two marbles) (see Fig. 21.6 of the lecture notes).

Empirically, we know there are only two kinds of elementary particles from the combinatorial point of view:

**bosons**: indefinitely many particles can assume a single identical one-particle state
   (examples: $\alpha$-particle, hydrogen atom, \(^4\text{He}\) atom);

**fermions**: all the indistinguishable particles must assume distinct one-particle states
   [Pauli’s exclusion principle] (examples: electron, proton, \(^3\text{He}\) atom).

There are $M$ distinguishable one-particle states and $n \leq M$ particles. Any particle can assume any one-particle state in $M$, if left alone. How many different combinations of one-particle states\(^3\) are possible, if particles are
(i) marbles,
(ii) $\alpha$-particles,
(iii) electrons?

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\(^1\)This is related to the partition of sets: https://en.wikipedia.org/wiki/Bell_number [thanks to J A Claes].

\(^2\)This is related to the number partition problem: https://en.wikipedia.org/wiki/Partition_(number_theory)

\(^3\)We later call such combinations specified microscopically microstates; do not confuse microstates and one-particle states.
2 [Events and sets]
(1) Let $\Omega = \{\omega_i\}$ be the sample space (= the totality of possible elementary events). Let $A\subset\Omega = \{\omega_1, \omega_3, \omega_4\}$. What is the meaning of the statement that event $A$ actually occurs?
(2) Let $A \subset \Omega$. Is event $A$ and event $\Omega$ statistically independent?

Perhaps, it is convenient to refurbish your knowledge of de Morgan’s law and the distributive law about $\cap$ and $\cup$ (see algebra of sets) before proceeding further.

(3) Find simple expressions for
   (i) $(A \cup B) \cap (A \cup B^c)$ (here $B^c = \Omega \setminus B$, the complement of $B$ in $\Omega$).
   (ii) $(A \cup B) \cap (B \cup C)$.
(4) Is $(A \cup B \cup C)^c = A^c \cap B^c \cap C^c$ true?
(5) There are three events $A$, $B$ and $C$ in the common $\Omega$. Illustrate the following events, using Venn diagrams:
   (i) No more than two events occur.
   (ii) At least two events occur.

3 [Elementary probability questions]
(1) Try to understand Q3-1 thoroughly (i.e., try to solve it first by yourself with sparingly consulting the solutions).
(2) Two fair coins are thrown but you cannot see them. You are told at least one coin exhibits a Head (H) and that if there is a coin exhibiting a Tail (T), you will be awarded $1,000. However, to participate in this game, you must pay a participation fee of $500. Will you still play the game, expecting some monetary gain?\footnote{Perhaps a discussion problem: there are two electrons and their spins, up or down, are measured. Let us consider the gambling problem with the coins H/T replaced with electrons U/D. What will you decide?}

4 [Law of large numbers]
Throwing a coin 1,000 times, you get 611 heads, so you suspect the coin is not fair. How rational is this conclusion from the point of view of the law of large numbers?

5 [Monte-Carlo estimate]
Design a ‘dart-throwing experiment’ to estimate $\sqrt{2}$. To obtain two digits for $\sqrt{2}$ with the failing rate of once in 100 trials, how many darts do you have to throw? An order of 100, 1000, or? Give a reasonable guess.

6 [Borel-Cantelli lemma related] [Discussion problem]
Let us repeat coin-tossing infinitely many times. Then, irrespective of the fairness of the coin (as long as both $H$ and $T$ are possible at all), we intuitively expect that we will observe infinitely many heads. In other words, we expect that with probability one we will not see
only finitely many heads.

The relevance of this statement to statistical thermodynamics is: for macrosystems what we actually observe with positive probability is determined by a set of macroscopically many microstates.\(^5\)

This should be intuitively obvious, but can you prove it from the ‘axioms’ of the probability: \(P\) is an additive set function whose range is \([0,1]\)?

(1) Let \(A_k\) be the event that the \(k\)-th trial (throwing) gives a head. What is the meaning of \(B_N = \bigcap_{k=N}^{\infty} A_k\)?

(2) In terms of \(B_N\) express the event \(F\) that only finitely many heads occur.

(3) Do you see \(B_N \subset B_{N+1}\)? This means

\[
P(F) = \lim_{N \to \infty} P(B_N). \tag{DH1.1}
\]

(4) Show

\[
P(B_N) = \prod_{k=N}^{\infty} (1 - P(A_k)). \tag{DH1.2}
\]

This implies \(P(F) = 0\).

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\(^5\)As noted above, ‘microstate’ is a microscopically described state of the system. For example, ‘half the spins in a magnetic lattice are up’ does not describe a microstate, since the spins of the atoms sitting at lattice points are not individually described; ‘the spins of the atoms sitting at lattice points \(\{x_i\}\) are up’ specifies a single microstate.
Homework 2 due at midnight on Feb 4 (M), 2019.
Submit to Kathryn: xueying8@illinois.edu.

You may discuss with your friends AFTER you have made due efforts of your own to solve the problems. I trust you. I wish you to fully understand the solutions when you submit your homeworks (and get the full credit).

No solution without your justification will get any credit.

1 [Elementary combinatorics]
(1) There are 5 distinguishable containers and 5 particles. Obtain the numbers of ways to distribute these particles for the cases:
   (i) protons,
   (ii) candies,
   (iii) hydrogen atoms.
(2) How many ways to put three (3) $^3$He atoms and three (3) $^4$He atoms in 5 distinguishable boxes? Ignore any energetic interactions among them.

2 [Sets and events]
Show the following statements:
(1) If two events $A$ and $B$ are statistically independent, then $A^c$ and $B^c$ are statistically independent as well.
(2) Any event $A$ and $\emptyset$ are statistically independent.

3 [Law of large numbers]
We wish to compute the following integral

$$I = \int_0^1 dx (1 - x^2)$$  \hspace{1cm} (DH1.3)

by the Monte-Carlo method. That is, we draw the graph of $(1 - x^2)$ and consider the area below it (between the graph and the $x$-axis in $[0,1]$) in the square $[0,1] \times [0,1]$ by peppering points uniformly on the square. To get $I$ within 1% relative error and with a failing rate of once in 500 trials, how many points $N$ do you need?

4 [Apparent aftereffect]
Shooter A hits the target with probability 0.8 and B with probability 0.4. They shoot simultaneously and one bullet hits the target. What is the probability that the bullet is due to B?

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For particles, interpret the containers as one-particle states as in Discussion.