You may discuss with your friends AFTER you have made due efforts of your own to solve the problems. I trust you. I wish you to fully understand the solutions when you submit your homeworks (and get the full credit).

No solution without your justification will get any credit.

1 [Elementary combinatorics]
(1) There are 5 distinguishable containers\(^1\) and 5 particles. Obtain the numbers of ways to distribute these particles for the cases:
   (i) protons,
   (ii) candies,
   (iii) hydrogen atoms.
(2) How many ways to put three (3) \(^3\)He atoms and three (3) \(^4\)He atoms in 5 distinguishable boxes? Ignore any energetic interactions among them.

Soln.
(1)
(i) This is a fermion case, so there is only 1 possibility: each proton is in each one-particle state.
(ii) This is a distinguishable (classic) case, so \(5^5 = 3125\) ways.
(iii) This is a boson case, so \(\binom{9}{5} = 9 \cdot 8 \cdot 7 \cdot 6/4 \cdot 3 \cdot 2 = 126\) ways.
(2) Since the distinct particles do not interfere statistically, and since we assume there is no physical interaction among the particles, we can simply superpose the results for distinct particles.
\(^3\)He are fermions and \(^4\)He are bosons. Therefore,
\[
\binom{5}{3} \times \binom{7}{3} = 10 \times 35 = 350
\]
ways.

2 [Sets and events]
Show the following statements:
(1) If two events \(A\) and \(B\) are statistically independent, then \(A^c\) and \(B^c\) are independent as well.
(2) Any event \(A\) and \(\emptyset\) are statistically independent.

Soln.
(1) We wish to show \(P(A^c \cap B^c) = P(A^c)P(B^c)\).
\[
P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - [P(A) + P(B) - P(A \cap B)]
\]
\(^1\)For particles, interpret the containers as one-particle states as in Discussion.
\[
1 - P(A \cup B) = 1 - [P(A) + P(B) - P(A)P(B)] = (1 - P(A))(1 - P(B)). \tag{DH1.2}
\]

(2) \( P(A \cap \emptyset) = P(\emptyset) = 0 = P(A)P(\emptyset) \). Or any event and \( \Omega \) are statistically independent as we have shown in Discussion, so, in particular, \( A^c \) and \( \Omega \) are statistically independent, so (1) gives what we want.

### 3 [Law of large numbers]

We wish to compute the following integral

\[
I = \int_0^1 dx \ (1 - x^2) \tag{DH1.4}
\]

by the Monte-Carlo method. That is, we draw the graph of \((1 - x^2)\) and consider the area below it (between the graph and the \(x\)-axis in \([0, 1]\)) in the square \([0, 1] \times [0, 1]\) by peppering points uniformly on the square. To get \( I \) within 1\% relative error and with a failing rate of once in 500 trials, how many points \( N \) do you need?

**Soln.**

Let \( \chi \) be the index of the set sandwiched between the \(x\)-axis and the graph of \( y = 1 - x^2 \) in the square \([0, 1] \times [0, 1]\) (the red region in Fig. DH1.1). Then, \( I = \langle \chi \rangle \).

Since Chebyshev’s inequality reads

\[
P \left( \left| \frac{1}{N} \sum_{i=1}^{N} \chi(x_i) - I \right| > \epsilon \right) < \frac{V}{\epsilon^2 N}, \tag{DH1.5}
\]
where $x_i$ is the location of the $i$th point landing on the unit square and $V$ the variance of $\chi(x_i)$. In our case, $\epsilon = 0.01I$ and the failure probability is $1/500$, so

$$\frac{V}{10^{-4}I^2N} \leq \frac{1}{500}. \quad \text{(DH1.6)}$$

We know $V = I(1 - I)$, so

$$N \geq 500 \times 10^4(1 - I)/I. \quad \text{(DH1.7)}$$

We know $I = 2/3$, so $N \geq 2.5 \times 10^6$.

4 [Apparent aftereffect]

Shooter A hits the target with probability 0.8 and B with probability 0.4. They shoot simultaneously and one bullet hits the target. What is the probability that the bullet is due to B?

**Soln.**

We need the conditional probability under the condition that one bullet hits the target. We may assume that the performance of the shooters is mutually statistically independent, so the expected probabilities of the relevant events are

- a: only A hits $0.8(1 - 0.4) = 0.48$,
- b: only B hits $(1 - 0.8)0.4 = 0.08$,
- ab: both hit $0.8 \times 0.4 = 0.32$,
- e: none hits $(1 - 0.8)(1 - 0.4) = 0.12$.

Therefore, event b under one bullet hitting the target must be the conditional probability of event b under the condition that a or b occurs: Thus, $P = 0.08/(0.48 + 0.08) = 0.143$. Notice that this is much smaller than 0.4.