We discuss basic thermodynamics (+ linear algebra we need for understanding elementary quantum mechanics).

1* [How to kill Thomson?]¹
In a very long thin cylinder is a gas (see Fig. DH5.1). We assume that the cylinder is diathermal and everything is performed under constant temperature in a uniform gravitational field. Initially,
(I) the cylinder is horizontal. Then,
(II) the cylinder is rotated to a vertical position (slowly). Due to the gravity the top portion becomes thin.
(III) We push in the piston till the pressure at the piston is the same as in (I). Then,
(IV) the system is rotated back to the original horizontal position with the piston position fixed relative to the cylinder.
(V) After the gas density becomes uniform, let us allow the system to do work till the pressure at the piston becomes identical to that in (I).

Thus, we have completed a cycle. Since, the work we do in (III) is less than that we gain from (V) → (I), we have killed Thomson!²

Is this OK?

[Hint] The work you obtain from this cycle is supplied by yourself through displacing the center of mass of the system.

¹Taken with slight modification from H. Tasaki Thermodynamics (Baifukan 2000).
²Of course, you should clearly recognize that what we now call Thomson’s principle was a special case of the original Clausius’ principle (according to him, heat at $T = \infty$ is work), and that Thomson failed to formulate thermodynamics.
2 [Adiabatic curves]
On the $PV$-diagram of any gas, two distinct adiabatic curves never cross. Demonstrate this. [This is almost a trivial question.]

Soln.
Look at the ‘triangular’ region surrounded by a high temperature isotherm and two adiabats in Fig. DH5.2.

![Diagram](image)

Figure DH5.2: Black curves are isotherms ($T$ const), and red curves are adiabats (actually, $S$ const). In this diagram, two adiabats cross, but such crossing never occurs.

If we go around this cycle in the clockwise direction.

$$-\oint PdV = -\text{ the area surrounded by these three curves} < 0,$$

which is the work gained by the gas, but it is negative. That is, it can perform work using a single heat bath, violating Kelvin’s law.

3 [Exact and not exact differentials]
Consider a function $f$ defined on a region $D \subset \mathbb{R}^n$. If its gradient is well defined on $D$, we say $f$ is (strongly) differentiable. We can write the first differential of $f$ as

$$df = \text{grad} f \cdot dx,$$  \hspace{1cm} (DH5.1)

where $x \in D$ is an $n$-vector (independent variables $x_1, \cdots, x_n$).

Let $C$ be any continuous curve in $D$ connecting $x_1$ and $x_2$. Then,

$$f(x_2) - f(x_1) = \int_C \text{grad} f \cdot dx$$  \hspace{1cm} (DH5.2)

does not depend on $C$. Thus, for any closed curve $C$ in $D$

$$\oint_C \text{grad} f \cdot dx = 0.$$  \hspace{1cm} (DH5.3)

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$^3$When we say a function is differentiable, it is always in this (strong) sense throughout these lectures.
If \( g = (g_1, \ldots, g_n) \) is a differentiable vector, the Gauss-Stokes-Green theorem tells us for any 2-surface \( A \)
\[
\int_A \sum_{i < j} (\partial_i g_j - \partial_j g_i) dx_i dx_j = \int_{\partial A} \sum_i g_i dx_i. \tag{DH5.4}
\]
Therefore, if the integral from point \( P \) to \( P' \) of \( \epsilon = \sum_i g_i dx_i \) is independent of the paths connecting these two points in a domain \( D \) (without a hole, or more precisely, contractible to a point), then
\[
\partial_i g_j = \partial_j g_i. \tag{DH5.5}
\]

**Remark.** If you know differential forms, you can say \( \epsilon \) is closed (\( d\epsilon = 0 \)) and there is a function \( g \) such that \( dg = \epsilon \) (i.e., \( \epsilon \) is exact; Poincaré’s lemma).

If \( f \) is twice differentiable (i.e., all the first order partial derivatives of \( f \) are differentiable), then for any 2-surface \( A \) in \( D \) such that \( \partial A = C \) (Gauss-Stokes-Green theorem)
\[
\oint_C \text{grad } f \cdot d\mathbf{x} = \int_A \sum_{i < j} \left( \frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i} \right) dx_i dx_j = 0. \tag{DH5.6}
\]
This is true for any closed curve \( C \) in \( D \), so we must conclude (called Young’s theorem) in \( D \)
\[
\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}. \tag{DH5.7}
\]
This is called a Maxwell’s relation in thermodynamics as we will study in detail later.\(^4\)

The first law of thermodynamics implies that the energy form \( dE \) is closed (so the formulas corresponding to (DH5.7) holds).

(1) Consider
\[
\epsilon = y^2 dx + 2x(y + 1) dy. \tag{DH5.8}
\]
(i) Check that this is not closed.\(^5\)
(ii) Integrate \( \epsilon \) along \( y = x^2 \) from the origin to (1, 1).
(iii) Integrate \( \epsilon \) along a part of a circle \( x^2 + (y - 1)^2 = 1 \) from the origin to (1, 1) counterclockwisely.

**Soln.**
(i) We check the symmetry (DH5.7):
\[
\frac{\partial y^2}{\partial y} = 2y, \quad \frac{\partial}{\partial x} 2x(y + 1) = 2(y + 1)
\]
so \( \epsilon \) is not closed.
Or more directly,
\[
d\epsilon = 2y dy \wedge dx + 2(y + 1) dx \wedge dy = [2(y + 1) - 2y] dx \wedge dy \neq 0. \tag{DH5.10}
\]
\(^{4}\)A 1-form \( \omega \) satisfying \( d\omega = 0 \) is called a closed form. Thus, an exact form is a closed form (i.e., \( d^2 = 0 \)). The converse is true on an orientable contractible domain (Poincaré’s lemma: if \( d\omega = 0 \), there is \( f \) such that \( df = \omega \)).
\(^{5}\)If you know differential forms, \( d\epsilon \neq 0 \) is what you have to show. That is, check something like (DH5.7).
(ii) Let us parameterize the curve as $x = t$ and $y = t^2$ ($t \in [0, 1]$). The integral reads
\[
\int_{y=x^2 \text{ for } x=0 \to 1} [y^2dx + 2x(y+1)dy] = \int_0^1 dt \left[ t^4 dt + 2t(t^2 + 1)2tdt \right] \tag{DH5.11}
\]
\[= \int_0^1 dt \left[ 5t^4 + 4t^2 \right] = 1 + 4/3 = 7/3 \approx 2.33. \tag{DH5.12}
\]

(iii) We set $x = \sin t$, $y = 1 - \cos t$ ($t \in [0, \pi/2]$):
\[
\int_0^{\pi/2} \left[ (1 - \cos t)^2 \cos tdt + 2\sin t(2 - \cos t) \sin tdt \right] \tag{DH5.13}
\]
\[= \int_0^{\pi/2} dt \left[ 4 - \cos t - 6 \cos^2 t + 3 \cos^3 t \right] \tag{DH5.14}
\]
\[= 2\pi - 1 - 6 \times \frac{\pi}{4} + 3 \times \frac{2}{3} = \frac{\pi}{2} + 1 \approx 2.57. \tag{DH5.15}
\]

(2) Consider
\[\epsilon = y^2dx + 2xydy. \tag{DH5.16}\]

(i) Check that this is closed.
(ii) Integrate $\epsilon$ along $y = x^2$ from the origin to $(1, 1)$.
(iii) Integrate $\epsilon$ along a part of a circle $x^2 + (y - 1)^2 = 1$ from the origin to $(1, 1)$. Needless to say, you expect the answer agrees with that to (ii).
(iv) It is easy to see $\epsilon = d(xy^2)$, so any integral of $\epsilon$ from the origin to $(x, y)$ is just $xy^2$ irrespective of the actual integration path.

**Soln.**

(i) We check the symmetry (DH5.7):
\[\frac{\partial y^2}{\partial y} = 2y, \quad \frac{\partial}{\partial x} 2xy = 2y \tag{DH5.17}\]
so $\epsilon$ is closed.

Or more directly,
\[d\epsilon = 2ydy \wedge dx + 2ydx \wedge dy = [2y - 2y]dx \wedge dy = 0. \tag{DH5.18}\]

(ii) Let us parameterized the curve as $x = t$ and $y = t^2$ ($t \in [0, 1]$). The integral reads
\[\int_{y=x^2 \text{ for } x=0 \to 1} [y^2dx + 2xydy] = \int_0^1 dt \left[ t^4 dt + 2t(t^2 + 1)2tdt \right] \tag{DH5.19}\]
\[= \int_0^1 dt \left[ 5t^4 \right] = 1. \tag{DH5.20}\]

\[^6\text{You can simply replace } y \text{ with } x^2 \text{ in the integral to get the answer more easily in this case.}\]
(iii) We set $x = \sin t$, $y = 1 - \cos t$ ($t \in [0, \pi/2]$):

$$
\int_0^{\pi/2} [(1 - \cos t)^2 \cos t \, dt + 2 \sin t (1 - \cos t) \sin t \, dt] = \int_0^{\pi/2} dt [2 - \cos t - 4 \cos^2 t + 3 \cos^3 t] = \pi - 1 - 4 \times \frac{\pi}{4} + 3 \times \frac{2}{3} = 1.
$$

(DH5.21) (DH5.22) (DH5.23)

(iv) $xy^2(x = y = 1) - xy^2(x = y = 0) = 1$, of course.

(3)* Let us prove that under constant volume the pressure of any material is independent of temperature. The heat form $dQ$ satisfies, according to the first law of thermodynamics,

$$
dQ = dE + PdV.
$$

(DH5.24)

This implies

$$
\left( \frac{\partial Q}{\partial T} \right)_V = \left( \frac{\partial E}{\partial T} \right)_V, \quad \left( \frac{\partial Q}{\partial V} \right)_T = P + \left( \frac{\partial E}{\partial V} \right)_T.
$$

(DH5.25)

Therefore,

$$
\frac{\partial^2 E}{\partial V \partial T} = \left( \frac{\partial P}{\partial T} \right)_V + \frac{\partial^2 E}{\partial T \partial V}.
$$

(DH5.26)

This implies $(\partial P/\partial T)_V = 0$. The result obviously contradicts the ideal gas law. Why?

Soln.

To go from (DH5.25) to (DH5.26) the closedness of $dQ$ is assumed, but we know heat is path dependent, so Young’s theorem (Maxwell’s relation) cannot be used.

4 [Very elementary questions]

(1) Aluminum has a density 2700 kg/m$^3$ at 300 K under 1 atm. Its isothermal compressibility $\beta$ is 0.01385 GPa$^{-1}$. Assuming the compressibility is constant, estimate the work needed to compress 1 kg of aluminum block from 1 atm to 500 atm?

(2) Ammonia gas under 1 atm at 300 K flows into a heating pipe with a flow speed of 41 cm$^3$/s. The pipe contains a 100 $\Omega$ resistor carrying an electric current of 50 mA. The flowing ammonia gas comes out at temperature 304.1 K. What specific heat can you observe from this (and obtain it in J/K·mol)?

(3) The internal energy of one mole of a gas is given by

$$
E = \frac{3}{2} RT - \frac{a}{V},
$$

(DH5.27)

where $a$ is a positive constant. Let us adiabatically freely expand this gas from volume $V_1$ to volume $V_2$. What is the final temperature $T_2$, if the initial temperature is $T_1$?

The parameter $a$ describes the effect of attractive interactions among gas particles. Is
your result consistent with this meaning of the parameter \( a \)?

**Soln.**

(1) The work \( W \) we must supply satisfies

\[
dW = -PdV = -P \left( \frac{\partial V}{\partial P} \right)_T dP
\]

According to the definition

\[
\beta = -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_T,
\]

and we may assume this to be constant, so we may use the following approximation:

\[
\left( \frac{\partial V}{\partial P} \right)_T = -\beta V_0,
\]

where \( V_0 \) is the volume under 1 atm, which is \((\text{in} \text{m}^3)\)

\[
V_0 = 1/2700.
\]

Therefore,

\[
W = \beta V_0 \int_{P_0}^{P_1} PdP = \frac{1}{2} \beta V_0 (P_1^2 - P_0^2) = \frac{1}{2} \times (0.01385 \times 10^{-9}) \times \frac{1}{2700} (5 \times 10^7)^2
\]

\[
= 6.4 \text{ J}.
\]

(2) Everywhere the pressure is constant, so the specific heat we obtain from this experiment is a specific heat under constant pressure \( C_P \). The flow rate in moles is

\[
41/(22.41 \times 10^3(300/273)) = 1.66 \times 10^{-3} \text{ mol/s}.
\]

The heat energy introduced in one sec is \( I^2R = 0.05^2 \times 100 = 0.25 \text{ W} \), and the temperature increase due to this is 4.09 K. Therefore,

\[
C_P = \frac{0.25}{1.66 \times 10^{-3} \times 4.09} = 0.0368 \times 10^3 = 37 \text{ J/K·mol}.
\]

(3) There is no input of heat nor work, so \( E \) must be constant:

\[
\frac{3}{2}RT_1 - \frac{a}{V_1} = \frac{3}{2}RT_2 - \frac{a}{V_2}.
\]

Therefore,

\[
T_2 = T_1 + \frac{2a}{3R} \left( \frac{1}{V_2} - \frac{1}{V_1} \right).
\]

We see the temperature decreases by expansion. Since volume expansion results in the increase of interparticle distances, the potential energy goes up, because the interaction is attractive. Therefore, the kinetic energy must go down, resulting in cooling. The effect is
actually small; doubling the volume may give you order of 0.1 K temperature decrease for usual gases.

5 [Conceptual review of superb importance].
The thermodynamic space of a macroscopic system in equilibrium is where we develop thermodynamics. This space is spanned by the thermodynamic coordinates. Give two or three important features of the thermodynamic coordinates.

Soln.
(1) Thermodynamic coordinates consist of internal energy $E$ and work coordinates.
(2) Work coordinates are extensive quantities and are observable/controllable solely by macroscopic mechanics and electrodynamics (not necessarily classical if macroscopic) (including relativity, if needed). Thus, all the coordinates are unquestionably observable with a macroscopic means.
(3) Thanks to (2) no thermal physics is needed to build thermodynamics so we can avoid vicious circle.
(4) Thanks to extensivity, thermodynamic states (equilibrium states) can be described uniquely (recall, for example, ice-water). This is especially important if you wish to describe phase transitions thermodynamically, because coexisting phases may share intensive quantities (say, $T$).

6 [Review of linear algebra rudiments]
You must be able to explain all the italicized technical terms (concepts/quantities) below.

(1) Let $A$ be a $N \times N$ matrix ($N \in \mathbb{N}^+$, positive integers) whose elements are all 1.
   (i) What is its determinant?
   (ii) What is its rank?
   (iii) Give all the eigenvalues of $A$.

Soln.
(i) Unless $N = 1$, the matrix has at least a pair of identical raws or columns, so it determinant must be zero: $\det A = 0$.
(ii) The rank of a matrix is the number of linearly independent column vectors. Since all the columns are identical, there is only 1 distinct column: rank $A = 1$. This means there are $N - 1$ zero eigenvalues.
(iii) There is only one non-zero eigenvalue that can be obtained trivially multiplying $A$ to a $N$-column vector consisting of 1:

$$A \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = N \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$  \hspace{1cm} (DH5.37)

Thus, there is one eigenvalue $N$ and the rest are $N - 1$ zero eigenvalues.
(2) Let $B$ be a $N \times N$ matrix ($N \in \mathbb{N}^+$, positive integers) whose off-diagonal elements are all 1, but their diagonal elements are 3. That is $B = A + [2, 2, \ldots, 2]$.

(i) What is its rank?
(ii) Give all the eigenvalues of $B$.
(iii) What is its determinant?

**Soln.**

(i) Since all the column vectors are linearly independent rank $B = N$.
(ii) Notice that the characteristic equation for $B$ must be identical to the characteristic equation for matrix $A$ with $\lambda$ being replaced by $\lambda - 2$. That is, $B$ has a single eigenvalue $N + 2$ and $N - 1$ eigenvalues being equal to 2.
(iii) Since $\det B$ is the product of all the eigenvalues, $\det B = (N + 2)2^{N-1}$. This result or the easy diagonalizability of $B$ is worth remembering (to make solvable examples).

(3) Let $A$ be a linear operator that has the following matrix representation for a particular orthonormal basis $\mathbf{o}$ of $\mathbb{R}^3$.

$$A = \begin{pmatrix}
3 & 1 & 1 \\
1 & 3 & 1 \\
1 & 1 & 3
\end{pmatrix}. \quad \text{(DH5.38)}$$

(i) Give all the eigenvalues of $A$.
(ii) What is its determinant?
(iii) Give normalized eigenvectors corresponding to the eigenvalues in (i). You must make these three vectors to make an ON basis $\mathbf{g}$ of $\mathbb{R}^3$.
(iv) Find an orthogonal matrix $U$ defining an orthogonal transformation that diagonalizes $A$.
(v) Confirm that the resolution of unity explicitly in terms of the eigenvectors.
(vi) Confirm the spectral decomposition formula for $A$ explicitly.

**Soln.**

(i) - (ii) we can read off the results from (2).
(i) 5, 2, 2 are the eigenvalues.
(ii) 20 [Of course you can use an elementary method to get this as $29 - 9$, but don’t do that after knowing the eigenvalues].
(iii) For this you might think you have to solve honestly

$$\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
\begin{pmatrix}
-2 & 1 & 1 \\
1 & -2 & 1 \\
1 & 1 & -2
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = 0, \quad \text{(DH5.39)}$$
but we already know a solution\textsuperscript{7}

\[ |5\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \]  

(DH5.40)

The eigenvectors belonging to 2 are the basis vector of the (hyper) plane perpendicular to

\[ |5\rangle, \]  

so, for example, we can choose (there are two eigenvectors belonging to 2, so we must
distinguish them with suffixes)

\[ |2_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad |2_2\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}. \]  

(DH5.41)

At least one of them, you can get honestly solving

\[
(A - 2I) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0,
\]

(DH5.42)

Since the rank of the matrix is 1, there are two linearly independent solutions. One may be

\[ x = 0 \text{ and } y + z = 0, \]

which is indeed \( |2_1\rangle \) after normalization. \( |2_2\rangle \) must be perpendicular to

\( |2_1\rangle \), so \((a, 1, 1)^T\) is easily guessed for it. This must also be perpendicular to \( |5\rangle \), so we can

fix \( a \). The result is \( |2_2\rangle \).

(iv) Summarizing the results so far, we have

\[
A \begin{pmatrix} 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}
\]

(DH5.43)

The matrix is an orthogonal (unitary) matrix, so its transposition is its inverse. That should

be obvious by its construction, but let us check:

\[
\begin{pmatrix} 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
\]

(DH5.44)

Therefore, (DH5.43) gives

\[
\begin{pmatrix} 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{pmatrix} A \begin{pmatrix} 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}
\]

(DH5.45)

\textsuperscript{7}There is a slight abuse of the notation here. The left-hand side is a ket, and the right-hand side is
its particular coordinate representation (or the componentwise expression with respect to the ON basis \( o \)).
This is not an uncommon abuse with respect to the original choice of the coordinate system = the original
ON basis \( o \).
That is, the needed unitary matrix is

$$U = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{pmatrix}. \quad (\text{DH5.46})$$

(v) Let us explicitly check the resolution of unity. The following should hold:

$$1 = |5\rangle\langle 5| + |2_1\rangle\langle 2_1| + |2_2\rangle\langle 2_2|.$$  \quad (\text{DH5.47})

Using our component representations of the kets (which correspond to column vectors), we can proceed as

$$\begin{align*}
\frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} (1, 1, 1) + \frac{1}{2} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} (0, 1, -1) + \frac{1}{6} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} (-2, 1, 1) \\
\frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} + \frac{1}{6} \begin{pmatrix} 4 & -2 & -2 \\ -2 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{align*} \quad (\text{DH5.48})$$

(vi) We expect the spectral decomposition as well:

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix} = |5\rangle\langle 5| + |2_1\rangle\langle 2_1| + |2_2\rangle\langle 2_2|.$$  \quad (\text{DH5.50})

should hold. Using our component representations of the kets, we can proceed as

$$\begin{align*}
\frac{5}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} (1, 1, 1) + \frac{2}{2} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} (0, 1, -1) + \frac{2}{6} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} (-2, 1, 1) \\
\frac{5}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 4 & -2 & -2 \\ -2 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}.
\end{align*} \quad (\text{DH5.51})$$