1 [Pomeranchuk effect]
The low temperature phase diagram of $^3$He is illustrated in Fig. DH13.1.

![Figure DH13.1: A schematic phase diagram of $^3$He.](image)

(1) Under constant pressure at low temperatures (below $\sim 0.3$ K), heating (the red arrow in Fig. DH13.1) solidifies $^3$He liquid. Which entropy is larger, the solid phase or the liquid phase? You must provide a supporting argument for your assertion.

(2) If solidification occurs by heating along the red arrow, does the density increase or decrease? You must provide a supporting argument for your assertion.

(3) $^3$He atom is a spin 1/2 particle, so we must take the magnetic order into account. The so-called spin exchange is the cause of spin spin coupling. In the usual magnet this exchange is mediated by the electron exchange among atoms, but in $^3$He it is mediated by the positional exchange of the atoms. In the solid phase atoms cannot move easily so positional exchange does not occur. In contrast, in the liquid phase, atoms exchange positions easily, so we must pay attention to the spin-spin coupling. What do you guess is the reason for the ‘strange’ phase diagram?

(4) If you increase the pressure of $^3$He reversibly and adiabatically across the liquid-solid phase transition line, what happens to the system temperature?

2 [Grand canonical approach to 1D van der Waals gas]
Let us study the 1D Kac model (see 25.10 in the notes) with the aid of the grand canonical approach. Here, to make the calculation easy, let us cheat a bit, replacing the interaction portion as the limiting form $-an$, where $n$ is the number density.\footnote{The honest approach must study the finite systems with the true Kac potential (25.10). After
(1) If there are \(N\) particles in the container of volume \(V\), the canonical partition function reads

\[
Z_N(V) = \frac{1}{h^N} \int_{(N-1)\sigma}^{V-\sigma} dx_N \cdots \int_{\sigma}^{\sigma_3-\sigma} dx_2 \int_0^{\sigma_2-\sigma} dx_1 \int dp_1 \cdots dp_N e^{-\sum_{i=1}^N p_i^2/2mk_BT+\mu N/k_BT},
\]

(DH13.1)

Here, \(\sigma\) is the particle hard core diameter. Check that, indeed, this is the right canonical partition function (with the afore-mentioned cheating). Then, actually compute the canonical partition function \(Z_N(V)\). [You virtually know the answer, if you consult Fig. 25.6 in the notes.]

(2) Using the result of (1) write down the grand canonical partition function. Since you cannot perform the summation over \(N\), you have only to write down the formula.

(3) The grand canonical partition function written down in (2) has the following structure:

\[
\Xi = \sum_{N=0}^M e^{F(N,V)},
\]

(DH13.2)

where \(M\) is the maximum number of particles we can push into volume \(V\). Use \(\Lambda \equiv 1 + (1/2) \log(2\pi mk_BT/h^2) + \mu/k_BT\) to simplify the result.

Show that if the temperature is sufficiently high, there is only one \(n = N/V\) that maximizes \(A(n) = F(N,V)/V\).

Also demonstrate that if the temperature is sufficiently low, there can be three extrema for \(A(n)\).

(4) What does the grand partition function look like, if \(n\) that maximizes \(A(n)\) are not unique?

(5) There is a text book\(^2\) which writes explicitly as follows:

\[
\Xi = e^{\beta PV} + e^{\beta' PV}.
\]

(DH13.3)

Here, we have assumed that \(A(n)\) have two maxima, and the two terms correspond respectively to the two maxima. Is this correct?

3 [Square lattice Ising model: mean field approach].\(^3\)

\(^2\)by Kardar
\(^3\)= Q27.2, because HW13 is related.
We have derived the fundamental equation (the consistency equation) for the starting point of the mean field approaches:

\[ \langle s_0 \rangle = \langle \tanh[\beta J(s_1 + \cdots + s_k)] \rangle \quad (\text{DH13.4}) \]

for an Ising model on a lattice with \( k \) nearest-neighbor spins around \( s_0 \). Let us take a lattice with \( k = 4 \) (square lattice, for example).

(1) Let us use the naivest approach as (26.20). Obtain the critical point \( T_c \) with this crude approximation.

Then, using a more accurate mean field theory explained in 27.2, we wish to exploit the fact that \( s^2 = 1 \). First, we expand \( \tanh \) in power series.

(2) Show that

\[ \tanh[\beta J(s_1 + \cdots + s_k)] = A(s_1 + s_2 + s_3 + s_4) + B(s_1s_2s_3 + s_1s_3s_4 + s_2s_3s_4 + s_1s_2s_4). \quad (\text{DH13.5}) \]

That is, any odd power of \( (s_1 + s_2 + s_3 + s_4) \) is written as a sum of \( (s_1 + s_2 + s_3 + s_4) \) and \( (s_1s_2s_3 + s_1s_3s_4 + s_2s_3s_4 + s_1s_2s_4) \).

(3) Determine \( A \) and \( B \) by setting \( s = \pm 1 \) so that (DH13.5) holds, or show that

\[ A = \frac{1}{8}(\tanh 4\beta J + 2 \tanh 2\beta J). \quad (\text{DH13.6}) \]

(4) Now, introducing (DH13.5) into (DH13.4), we get the following equation

\[ \langle s_0 \rangle = A(s_1 + s_2 + s_3 + s_4) + B(s_1s_2s_3 + s_1s_3s_4 + s_2s_3s_4 + s_1s_2s_4). \quad (\text{DH13.7}) \]

\( \langle s_0 \rangle = \langle s_1 \rangle = \cdots = m \) is the magnetization per spin, so (DH13.7) reads

\[ m = 4Am + 4Bm^3. \quad (\text{DH13.8}) \]

Notice that up to this point there is NO APPROXIMATION, but, unfortunately, we cannot solve (DH13.8). Now, let us introduce the approximation

\[ \langle s_1s_2s_3 \rangle = m^3. \quad (\text{DH13.9}) \]

Then, our ‘approximate’ mean field equation is

\[ m = 4Am + 4Bm^3. \quad (\text{DH13.10}) \]

What is the condition that determines the phase transition? [Hint. At what value of \( A \) is there a bifurcation?]

\(^4\text{Recall ‘bifurcation’ implies the change of the number of (real) roots.}\)
4. [Simple 1D renormalization (decimation)]

Let us study 1-Ising model with the aid of *decimation* illustrated in Fig. DH13.2. This procedure thins the spins through summing over a subset of spins, keeping the rest fixed.

The original canonical partition function reads (here, $K = \beta J$)

\[
Z = \sum_{\sigma} \cdots e^{K s_{-1} \sigma_0 + \sigma_0 s_1} \cdots, \tag{DH13.11}
\]

where spins at the even lattice positions are written as $\sigma$.

Figure DH13.2: Decimation: we sum over the red spins.

(1) Summing over all $\sigma$ states, the original $Z$ now reads

\[
Z = C \sum_{s} \cdots e^{K'(s_{-1}s_1)} \cdots, \tag{DH13.12}
\]

where $C$ is a constant. This can be understood as ($C$ times) the canonical partition function of the chain consisting of the remaining odd lattice spins $\{s_i\}$. Actually, each term in the sum is a product of the factors (proportional to)

\[
\sum_{\sigma_0 = \pm 1} e^{K(s_{-1} \sigma_0 + \sigma_0 s_1)}, \tag{DH13.13}
\]

so to compute $K'$ we should study

\[
\sum_{\sigma_0 = \pm 1} e^{K(s_{-1} \sigma_0 + \sigma_0 s_1)} = e^{A + K's_{-1}s_1}, \tag{DH13.14}
\]

where $A$ is a constant (and $C$ is the product of $e^A$s). Find $A$ and $K'$.

(2) Since we do not care for $C$, the result of (1) may be understood as the transformation of the system Hamiltonian from $H$ to $H' = \mathcal{D}H$ (decimation transformation):

\[
H = \sum_{i \in \mathbb{Z}} K s_i s_{i+1} \rightarrow H' = \mathcal{D}H = \sum_{i \in \mathbb{Z}/2} K' s_i s_{i+2}. \tag{DH13.15}
\]
The macroscopic observables of a very big system governed by $H$ should be understood from the behavior of the system governed by $D^n H$ for large $n$. Using this observation, show that there is no finite temperature phase transition in 1-space.
Homework 13 due 9 am on April 30 (Tu), 2019.

Submit to compass2g

You may discuss with your friends AFTER you have made due efforts of your own to solve the problems. I trust you. I wish you to fully understand the solutions when you submit your homeworks (and get the full credit).

No solution without your justification will get any credit. You must know how to write proper reports. Writing only formulas is totally unacceptable; your solution must read as a proper English composition.

As to the use of TeX: It was announced that from week 10 use of TeX (of some version) would be strictly imposed. The purpose is that you learn how to write mathematics properly, so if proper math orthography would be met, anything, including extremely neat hand writing, will be accepted. You must use proper aligning of formulas, correct punctuations, and correct fonts,\(^5\) etc., even with handwriting. Errors in math orthography will be penalized (but at most 30\% of the total score). Except for punctuations most requirements will be automatically satisfied, if you use (La)TeX.\(^6\) With Words you will have to struggle to meet the requirement. Handwriting is strongly discouraged.

You may send me (yoono@illinois.edu) TeX questions like: how to write/program “…”?

1. [Elementary questions]
(1) The density of a gas is usually less than the liquid of the same substance. Why?
(2) Does a triple point correspond to a single thermodynamic state?
(3) At the triple point (solid-gas-liquid coexistence; cf. Fig. 24.1 in the notes) of a pure substance, three coexistence curves, the LG, GS and LS curves, meet at the tricritical point on the \(PT\)-plane (\(P\) the vertical axis). Which curve is the steepest, and which the least steep around the triple point? You must justify your answers.
(4) When a magnetic field is applied in the \(z\)-direction, a phase transition from phase I to phase II occurs. What can you say about the change of the magnetization?

\(^5\)italicized or not in particular; basically, all the formulas are in italic and all the ordinary English sentences are in upright

\(^6\)Although I have no intention to recommend my own macros, if you use something like them, then you need only to be able to ‘read formulas loud.’ That is why I posted all the source files; in most cases you can copy some parts of them with modifications.
2 [Phase coexistence]
There is a mixture of two chemicals A and B. This mixture can have 4 phases, solid $(S)$, liquid I $(L)$, liquid II $(L')$ and gas $(G)$ phases. In the gas phase A and B mix freely, but the solid phase consists only of A (i.e., B cannot get into the solid phase). Can you have a quadruple point $(T_Q, P_Q)$ where these 4 phases coexist? You may assume any reasonable functional equation can be solved.

3. [2-Ising model on the honeycomb lattice; mean-field approach]
Let us consider a 2-Ising model on the honeycomb lattice whose coupling constant is $J$. Assume there is no magnetic field.
(1) Find the equation corresponding to (26.19) [consistency equation] in the notes.
(2) Find $T_c$ with the aid of the approximation corresponding to (26.20) [the naivest approximation] in the notes.
(3) Then, using a more accurate mean field theory explained in 27.2 of Discussion 13.3, compute $T_c$. Which $T_c$ obtained by (2) or by this question should be lower? Is your result consistent with your expectation?