We will mainly discuss low temperature ideal quantum systems. The explanations of the problems may be read independently (so the same comments and formulas are often repeated).

1 [Zeeman splitting]
An electron in the outer shell of an ion has magnetic moment of one Bohr magneton $\mu_B$. In a magnetic field $B$, this outer shell state splits into two energy levels with $E = E_0 \pm \mu_B B$ (down or up states; the up state is more stable, because it is parallel to $B$). The magnetization $M = \mu_B(n_u - n_d)$, where $n_u$ (resp., $n_d$) is the occupation number of the up (resp., down) state. You can assume that electrons do not interact with each other.

(1) Find $\langle M \rangle$ and $\langle N \rangle$ with a grand canonical ensemble, assuming that there are $N$ ions.

(2) Give the average magnetization when the outer shell contains exactly one electron, and compare it with the result for $\mu = E_0$ in (1).

Semi-quantitative questions
The following problems are semi-quantitative questions about low temperature ideal systems.

In answering this type of questions recall some obvious facts:
(i) How the (one-particle) energy levels of a single particle confined in a box changes, if its volume is changed: If squished, the level spacings widen, and if expanded, they

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2Actually the field is in the $z$-direction, and $B$ is its $z$-component.

3[This is a comment for those who are extremely careful about logic.] What is written here is the way the usual books ask the question, but, strictly speaking, it is not the usual GCE explained in the books (and in my lecture notes), because $B$ is also fixed ($M$ is not fixed), so $E - ST - \mu N - MB$ is the Legendre transformation, instead of the usual $E - ST - \mu N$. It is, HOWEVER, still $PV = ST + \mu N + BM - E$ is the corresponding thermodynamic potential.

What is the conclusion of this precise argument? You must be careful when you use the Gibbs relation. If $B$ is fixed instead of $M - d(PV) = -SdT - PdV - Nd\mu - Mb$ rather than $-d(PV) = -SdT - PdV - Nd\mu + BdB$.

In practice, you may regard the GCE as the ensemble for which all the extensive quantities but $V$ are allowed to change by fixing their corresponding conjugate intensive variables.
shrink (Fig. DH11.1).

(ii) If particles’ energies shift with the energy levels (as if the particles just perch on the shifting energy levels), then the system changes adiabatically and reversibly; the system entropy stays constant.\(^4\)

(iii) Try to use thermodynamics as much as possible. Recall \(E = TS - PV + \mu N\).\(^5\)

(iv) \(PV = 2E/3\) is universal.\(^6\)

(v) Recall how to ‘guess’ one-particle state density: \(D_t\) (21.14 in the notes); often dimensional analysis works.

(vi) Very low temperature features of average occupation numbers: fermions are with sharp cliffs (of width \(\sim k_B T\)); bosons below \(T_c\) are with \(\mu = 0\).

3. [Pressure changes under various conditions]
Assume that the particles do not interact, and answer the following questions for both ideal bosons and ideal fermions (both without any internal degree of freedom). You may assume that the initial temperature is sufficiently low (e.g., well below \(T_c\) for bosons). If your argument is sufficiently convincing in terms of elementary physics, you need not demonstrate your answers with the aid of formulas, but I strongly recommend you to check your intuitive answer using thermodynamics and/or statistical mechanics.

(1) The pressure \(P\) is increased under constant internal energy. Does the temperature of the gas increase? Assume that the initial temperature is sufficiently low (below \(T_c\) for bosons).

(2) The pressure \(P\) is increased under constant entropy. Does the temperature of

\(^4\)This is the so-called quantum adiabatic change. Quantum adiabatic change is an example of thermodynamic adiabatic changes. However, it is an extremely special (unrealizably special, I should say) case. Such a process certainly preserves entropy, but it is far from necessary.

\(^5\)As long as we keep the energy origin to be the one-particle ground state. Read, however, an important warning in 7.

\(^6\)As long as the system is 3-dimensional and the kinetic energy is given by \(mv^2/2\).
the gas increase?
(3) The pressure $P$ is increased under constant volume. Does the temperature of the gas increase?

5 [Quasistatic adiabatic expansion]
There is an ideal gas in a box with a piston. The whole system is thermally isolated. We double the volume in a quasiequilibrium fashion. Let the initial pressure be $P_i$ and the initial temperature $T_i$.

Suppose the ideal gas consists of noninteracting fermions without any internal excitations.
(1F) Obtain the final pressure $P_f$ in terms of the initial pressure $P_i$.
(2F) If $T_i = 0$, what is the final temperature $T_f$?
(3F) Suppose $T_i > 0$. What can you say about $T_f$?

Next, let us assume the ideal gas is a non-interacting bose gas.
(1B) Obtain the final pressure $P_f$ in terms of the initial pressure $P_i$.
(2B) If $T_i = 0$, what is the final temperature $T_f$?
(3B) Find $T_f$ in terms of $T_i$.
(4B) Let $N_{0i}$ be the number of particles in the condensate initially. Is $N_{0f}$, the final number of particles in the condensate, larger or smaller than $N_{0i}$ or unchanged?

6 [Isothermal compression]
There is a cylinder with a piston. It contains $N$ identical particles and is maintained at a constant temperature $T$.
Fermion case
(F1) Suppose the system is maintained at $T = 0$. The volume is halved reversibly. The initial energy per particle is $e_i$. What is the final energy per particle $e_f$ in terms of $e_i$?
(F2) What is the final and initial pressure ratio $P_f/P_i$ at $T = 0$?

Boson case
(B1) Suppose the condensate density is positive under the initial temperature. Is the condensate density positive even after reversible isothermal compression $V \rightarrow V/2$?
(B2) What is the final and initial pressure ratio $P_f/P_i$?

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\(^7\)Discussed in a lecture, but let us record it here with more details.
7 [Constant energetic compression]
A box of volume $V$ with a piston is filled with $N$ indistinguishable ideal gas atoms at temperature $T_i$. The final equilibrium state is obtained by halving the volume and removing the heat to maintain the total energy constant. That is, the final state is volume $V/2$ and internal energy $E$. Let $T_f$ be the final temperature.

Suppose $N$ particles are identical spinless bosons. Assume that the initial temperature is sufficiently low so there is a Bose-Einstein condensate.
(B1) Obtain (or write down/copy) the equation to find the number $N_0$ of atoms in the condensate.
(B2) Is $T_f < T_i$, $T_f = T_i$ or $T_f > T_i$?
(B3) Does the number of particles in the condensate increase or decrease?

Suppose $N$ particles are identical spin 1/2 fermions.
(F1) Find the final pressure $P_f$.
(F2) Is there any positive lower bound for the initial temperature for this process to be feasible?
(F3) Is $T_f < T_i$, $T_f = T_i$ or $T_f > T_i$?

8 [Adiabatic sudden expansion]
There is an ideal gas in a thermally isolated box with a piston. We double the volume suddenly by pulling the piston out very rapidly. Let the initial pressure be $P_i$ and the initial temperature be $T_i$.

Suppose the ideal gas consists of noninteracting fermions without any internal excitations.
(F1) Obtain the final pressure $P_f$ in terms of the initial pressure $P_i$.
(F2) $T_i < T_f$, $T_i = T_f$ or $T_i > T_f$?

Next, let us assume the ideal gas is a non-interacting bose gas.
(B1) Obtain the final pressure $P_f$ in terms of the initial pressure $P_i$.
(B2) Let us assume that the system is below the BEC temperature even after expansion. What is the final temperature $T_f$?
(B3) The initial temperature $T_i$ is below $T_c$. After expansion, the temperature $T_f$ was exactly at the critical temperature (after expansion $T_{cf}$). What is the initial temperature $T_i$ in terms of the $T_c$ (before expansion $T_{ci}$).
9 [Condensation in 2D-harmonic trap].

There is a 2-dimensional harmonic trap $U = (1/2)\alpha x^2$, where $x$ is the distance from the origin, and $\alpha$ is a positive constant. We know the single particle energy levels in this trap are denoted as

$$\epsilon = \hbar \omega (1 + n_1 + n_2), \quad (\text{DH11.1})$$

where $n_1, n_2 \in \mathbb{N} = \{0, 1, 2, \ldots\}$, and $\omega$ is a positive constant. In the following to make the ground state energy to be zero, the zero-point energy is ignored.

(1) The density of state $D(\epsilon)$ is the number of states with energy between $\epsilon$ and $\epsilon + d\epsilon$. Therefore, we know

$$\int_{0}^{\epsilon} d\epsilon' D(\epsilon') = \sum_{n_1+n_2 \in [0, \epsilon/\hbar \omega]} 1. \quad (\text{DH11.2})$$

Noting that the sum on the right-hand side is essentially the area of the shaded triangle in Fig. DH11.2, obtain $D(\epsilon)$.

![Figure DH11.2: The relation between $n_1$, $n_2$ and $\epsilon/\hbar \omega$](image)

(2) Is there an Einstein condensation in this 2D trap? [Hint. Compute $N_1$ and study whether it is finite or not for $\mu = 0$. Mimic our argument in 3D free space. You must choose the energy origin to be the ground state.]
Homework 11 due 9 am on April 18 (Th), 2019.\footnote{This is not an error; since the Midterm deadline is 16th, I shift the deadline for two days.}
Submit to compass2g

You may discuss with your friends AFTER you have made due efforts of your own to solve the problems. I trust you. I wish you to fully understand the solutions when you submit your homeworks (and get the full credit).

No solution without your justification will get any credit.

\textbf{As to the use of TeX:} It was announced that from week 10 use of TeX (of some version) would be strictly imposed. The purpose is that you learn how to write mathematics properly, so if proper math orthography would be met, anything, including extremely neat hand writing, will be accepted. You must use proper aligning of formulas, correct punctuations, and correct fonts,\footnote{Italicized or not in particular; basically, all the formulas are in italic and all the ordinary English sentences are in upright} etc., even with handwriting. Errors in math orthography will be penalized (but at most 20\% of the total score). Except for punctuations most requirements will be automatically satisfied, if you use (La)TeX.\footnote{Although I have no intention to recommend my own macros, if you use something like them, then you need only to be able to ‘read formulas loud.’ That is why I posted all the source files; in most cases you can copy some parts of them with modifications.} With Words (and handwriting) you will have to struggle to meet the requirement.

You may send me (yoono@illinois.edu) TeX questions like: how to write/program “…”?

1 \textbf{[Finite $T$ correction for fermion chemical potential in $D$-space].}

(1) Assuming that the dispersion relation (the momentum-energy relation) is $\varepsilon = |\mathbf{p}|^2/2m$, find the relation among $E$, $P$ and $V$ in $D$-dimensional space of an ideal gas system. [Mimic the derivation of $PV = (2/3)E$.]

(2) What is the ratio $E/\mu(0)N$ at $T = 0$, if all the particles are identical fermions?

(3) For fermions we can conclude (under constant $V$ and $N$) that

$$E = E_0 + \frac{1}{2} \alpha_D NT^2 + o[T^2],$$

(DH11.3)

where $\alpha_D > 0$ is a ($D$-dependent) constant. Using this fact, compute the correction to the Fermi energy $\mu(T)$ to order $T^2$. [Fully use thermodynamics.]
(4) For bosons below $T_c (D \geq 3)$, find the temperature dependence of the non-condensate $N_1$.

2 [Adiabatic free expansion]
There is a cylinder with a piston. It contains $N$ identical particles and is thermally isolated. The volume of the cylinder is suddenly expanded (by pulling the piston out a bit) by 10%.

We wish to know what happens after the system equilibrates. Let $P_i$ ($P_f$) be the initial (final) pressure and $T_i$ ($T_f$) be the initial (final) temperature.

Fermion case
(F1) Find $P_f/P_i$.
(F2) Which is larger, $T_i$ or $T_f$? Explain your answer qualitatively in plain terms.

Boson case
(B1) Find $P_f/P_i$.
(B2) What happens, qualitatively, to the BEC (= Bose-Einstein condensation) temperature $T_c$? Explain your answer in plain terms intuitively.
(B3) Suppose $T_i = T_c$ for the initial system. Does the system maintain BEC after expansion?

3. Consider a quantum ideal gas (fermion and boson cases separately, if different). No hand-waving argument will be accepted. [Hint: Use thermodynamics (as much as possible to save your time) and $PV = 2E/3$ in this problem.]
(1) The volume is increased under constant temperature. Does the entropy increase? You must demonstrate your result without any hand-waving argument. Notice that thermodynamics alone cannot answer this question.
(2) You wish to decrease the temperature while keeping the pressure. How do you have to change the system volume?