1 Configuration space

Let $L$ be the whole lattice on which we consider a statistical mechanical system. At each $x \in L$ (lattice point) there is a set of ‘site states’ $\Omega_x$, which is a finite set.

Then, the configuration space $\Omega$ is $\Omega = \prod_{x \in L} \Omega_x$. $\xi \in \Omega$ is a configuration (state) of the system defined on $L$, and $\xi_x \in \Omega_x$. Often, not all the elements in $\Omega$ are allowed, so $\xi$ must satisfy some conditions. In the original text $\Omega$ is defined explicitly with the constraints such as $\xi_\Lambda (\xi$ on $\Lambda \subset L)$ is in $\Omega_\Lambda \subset \Omega$. $\Omega$ is a direct product of $\Omega_x$, and the latter is finite, so $\Omega$ is compact\(^1\) wrt the direct topology according to Tychonoff’s theorem (See Simon I100 Th 2.7.1).

For any $S \subset L$, $\Omega_S$ is compact. Observables are (continuous) functions defined on $\Omega (C(\Omega))$. $A \in (\Omega_\Lambda)$ is, strictly speaking, not defined on $C \equiv C(\Omega)$, but we interpret appropriately as a function on $C(\Omega)$ appropriately.

In the original the restriction map $\alpha_\Lambda : \Omega \to \Omega_\Lambda$ is explicitly defined and observable on $C$ is written explicitly as $A \circ \alpha_\Lambda$. $C$ is a Banach space.

The probability measures = (thermodynamic) states on $\Omega$ is a convex compact subset of the dual space $C^*$.\(^2\) $A \in C$ is a microscopic expression of the corresponding macroobservable $\mu(A)$ for a state $\mu$.

An interaction $\Phi$ is a real function on $\cup \Omega_\Lambda$, where the set join is taken over all the finite subset $\Lambda \subset L$. Usually, we impose for each $x \in L$

$$|\Phi|_x = \sum_{\Lambda \ni x} \frac{1}{|\Lambda|} \sup_{\xi \in \Omega_\Lambda} |\Phi(\xi)| < +\infty$$  \hspace{1cm} (1.1)

or

$$\|\Phi\|_x = \sum_{\Lambda \ni x} \sup_{\xi \in \Omega_\Lambda} |\Phi(\xi)| < +\infty.$$  \hspace{1cm} (1.2)

Here, the summation is over all the subsets of $L$ containing the lattice point $x$.

\(^1\)Compact\] A topological space $T$ is compact if its any open cover (need not be countable) contains a finite open cover. In $T$ any infinite set has an accumulation point. The topological space satisfying this condition is called countably compact. Thus, compact implies countably compact. However, the converse is not true. However, if $T$ has a countable basis (i.e., if $T$ satisfies the second countable axiom), then we can always choose a countable open cover of $T$. A necessary and sufficient condition for $T$ to be countably compact is that any countable open cover has a finite open cover. Thus, if a space is second countable (or metric or metrizable) space, then a necessary and sufficient condition for it to be compact is that any infinite set has an accumulation point.

Tychonoff’s theorem states that the direct product of compact spaces is again compact wrt the direct product topology. It can be demonstrated by the standard diagonal argument.

\(^2\)Convexity is obvious; its compactness (wrt the weak or vague topology: see Simon I237) follows from the compactness of $\Omega$.  

1
2 Gibbs ensemble

The Gibbs ensemble for the region $\Lambda \subset L$ and the interaction $\Phi$ is the probability measure $\mu_\Lambda$ on $\Omega_\Lambda$ defined by

$$\mu_\Lambda(\xi) = \frac{1}{Z_\Lambda} \exp[-U_\Lambda(\xi)], \quad (2.1)$$

where

$$Z_\Lambda = \sum_{\xi \in W_\Lambda} \exp[-U_\Lambda(\xi)], \quad (2.2)$$

and

$$U_\Lambda(\xi) = \sum_{x \in \Lambda} \Phi_x|_\Lambda \quad (2.3)$$

with $\Phi_x$ being basically the interaction energy assigned to lattice point $x$:

$$\Phi_x|_\Lambda(\xi) = \sum_{x \in X \subset \Lambda} \frac{1}{|X|} \Phi(\xi_X) \quad (2.4)$$

First of all, quite generally, if we can consider the existence of a limit of a sequence of marginal measures.

Let $\{M(n)\}$ be a sequence of subsets of $L$ such that $M(n) \to L$. We can define $\mu_{M(n)}$ defined on $\Omega_{M(n)}$. Then, we can choose a subsequence $\{M(n')\}$ such that

(i) for any finite subset $\Lambda \subset L \lim_{n' \to \infty} \mu_{M(n')}$ restricted on $\Lambda$ converges to $\rho_\Lambda$ (i.e., $\alpha_{\Lambda,M(M')} \mu_{M'} \to \rho_\Lambda$).

(ii) There is a measure $\rho$ defined on $\Omega$ such that $\rho_\Lambda = \alpha_{\Lambda,\rho}$.

Remark. Here $\alpha_{A,B}$ for $A \subset B$ implies the restriction. Therefore, for a measure defined on $B \alpha_{A,B} \mu$ implies the marginal measure for the configuration on $A$:

$$\alpha_{A,B} \mu(\xi_A) = \sum_{\xi|_A = \xi_A} \mu(\xi). \quad (2.5)$$

Proof: Notice that $\alpha_{A,M} \mu_M$ is an infinite set of probability measures on $\Omega_A$, so it is a compact set. Thus, there must be an accumulation point. Thus, (i) is almost trivial. (i) implies $\alpha_{A,M} \rho_M = \rho_\Lambda$, if $\Lambda \subset M$. Thus, for all finite $\Lambda \subset L$, $\{\rho_\Lambda\}$ makes a family of consistent measures. Then, Kolmogorov’s consistency theorem (Simon I296) implies (ii).

$\rho$ thus obtained is the thermodynamic limit of $\mu_M$.

Definition of Gibbs state $\sigma$

If for any finite $\Lambda \subset L \alpha_\Lambda \sigma$ may be obtained by a suitable average of the Gibbs ensemble with boundary contributions over the state outside $\Lambda$, we say $\sigma$ is a Gibbs state. More precisely, $\sigma$ is a Gibbs state with the interaction $\Phi$ if we can write with the aid of a probability measure $\tau$ for the ‘external configurations $\eta$ (configuration on $L \setminus \Lambda$)

$$\alpha_\Lambda \sigma(\xi_\Lambda) = \sum_{\eta \in \Omega_{L \setminus \Lambda}} \tau(\eta) \mu_{\Lambda,\eta}(\xi_\Lambda), \quad (2.6)$$
where $\mu_{\Lambda, \eta}$ is the canonical measure on $\Lambda$ with the external configuration $\eta$:

$$
\mu_{\Lambda, \eta}(\xi_{\Lambda}) = \frac{1}{Z_{\Lambda, \eta}} \exp \left( -U_{\Lambda}(\xi_{\Lambda}) - B(\xi_{\Lambda} \cup \eta) \right). \quad (2.7)
$$

Here, $B$ is the interaction between $\Lambda$ and its outside state.

We wish to demonstrate that if $\rho$ is a thermodynamic limit of Gibbs ensemble $\mu_{\Lambda}$ (recall that in this case no boundary condition is specified), then $\rho$ is a Gibbs state with the interaction $\Phi$.

Using the definition (2.1), for $\Lambda \subset M$

$$
[\alpha_{\Lambda, M} \mu_{M}](\xi_{\Lambda}) = \sum_{\eta \in \Omega_{M \setminus \Lambda}} \mu_{M}(\xi_{\Lambda} \cup \eta) \quad (2.8)
$$

$$
= \sum_{\eta \in \Omega_{M \setminus \Lambda}} \frac{1}{Z_{M}} \exp \left( -U_{\Lambda}(\xi_{\Lambda}) - U_{M \setminus \Lambda}(\eta) - B(\xi_{\Lambda} \cup \eta) \right) \quad (2.9)
$$

$$
= \sum_{\eta \in \Omega_{M \setminus \Lambda}} [\alpha_{M \setminus \Lambda, M} \mu_{M}](\eta)\mu_{\Lambda, \eta}(\xi_{\Lambda}) \quad (2.10)
$$

where

$$
\mu_{\Lambda, \eta}(\xi_{\Lambda}) = \frac{1}{Z_{\Lambda, \eta}} \exp \left( -U_{\Lambda}(\xi_{\Lambda}) - B(\xi_{\Lambda} \cup \eta) \right). \quad (2.11)
$$

Now, we wish to replace $M$ with an increasing sequence $\{M(n')\}$.

(1) Precisely speaking $B$ in (2.11) can depend on $M$, but $B$ converges to a limit in the $M \to L$ limit.

(2) Following the same logic used to demonstrate the existence of $\rho$, we can show that

$$
[\alpha_{M \setminus \Lambda, M} \mu_{M}](\eta) \to \alpha_{L \setminus \Lambda} \rho = \rho_{L \setminus \Lambda}. \quad (2.12)
$$

Therefore, (2.10) reads

$$
[\alpha_{\Lambda, M(n')} \mu_{M}(n')](\xi_{\Lambda}) \to \sum_{\eta \in \Omega_{L \setminus \Lambda}} \rho_{L \setminus \Lambda}(\eta)\mu_{\Lambda, \eta}(\xi_{\Lambda}) \quad (2.13)
$$

This means that a thermodynamic limit $\rho$ of $\mu_{M}$ is a Gibbs measure.

Actually, its converse is also true: if $\rho$ is a Gibbs measure, then we can introduce an appropriate boundary terms $B$ and interpret it as a Gibbs state.

This implies that fora Gibbs state $\sigma$, we can always approximate

$$
\frac{[\alpha_{M} \sigma](\xi_{\Lambda} \cup \eta)}{\sum_{\xi \in \Omega_{\Lambda}} [\alpha_{M} \sigma](\xi_{\Lambda} \cup \eta)} \approx \frac{1}{Z_{\Lambda, \eta}} \exp \left( -U_{\Lambda}(\xi_{\Lambda}) - B(\xi_{\Lambda} \cup \eta) \right) \quad (2.14)
$$

uniformly for large enough $M$. Thus we can summarize:

**Theorem 1.8** A probability measure $\sigma$ on $\Omega$ is a Gibbs state iff for each finite $\Lambda \subset L$ the conditional probability that $\xi|_{\Lambda} = \xi_{\Lambda}$ when $\xi|_{L \setminus \Lambda} = \eta$ is given by $\mu_{\Lambda, \eta}(\xi_{\Lambda})$ (2.7).

Notice that in (2.6) $\tau = \alpha_{L \setminus \Lambda} \sigma$. In particular, a vague limit of Gibbs states is a Gibbs state.
We have learned any thermodynamic limit of $\mu_{\Lambda,\eta|\Lambda}$ is a Gibbs state. Let $K$ be the totality of the states obtained this way. Then it is actually the totality of the Gibbs states with the interaction $\Phi$.

3 Algebra at infinity

Let $C_M$ be the algebra of functions $A \circ \alpha_M$ where $A \in C(\Omega_M)$. Since $\alpha_M$ is defined on the whole $L$, but ‘pays attention’ only to the state on $M$, the observable $A \circ \alpha_M$ is defined on the whole $L$, but the ‘observation’ is restricted only on $M$. Thus $C_M$ is the totality of observables completely determined by the configuration on $M$ only.

For a probability measure $\sigma$ on $\Omega$ $\pi(A)$ denotes a class of $A$ in $L^\infty(\Omega,\sigma)$ (essentially bounded on $\Omega$ wrt $\sigma$; i.e, on any $\sigma$-positive set $|A|$ is bounded.

\[ B_\sigma = \cap_{\Lambda \subset L,|\Lambda|<\infty} \left[ \bigcup_{M \subset L \setminus \Lambda,|M|<\infty} \pi(C_M) \right] \]  

(3.1)

is called the algebra at infinity associated with $\sigma$. The overline implies ‘closure.’ Take a finite domain $\Lambda$ on the lattice. Then we study the totality of essentially bounded observables that are completely determined by the configurations on the domain outside this finite set. Then, that outside $\cap$ chooses the limit $\Lambda \rightarrow L$. Thus, $B_\sigma$ is the totality of the observable that is determined by the configurations (events) far away (and essentially bounded wrt to $\sigma$).

If $B_\sigma$

4 Non-extremality

Let $\sigma_{L\setminus \Lambda,\eta}$ be the measure on $\Omega_{L\setminus \Lambda}$ with the condition on $\Lambda$: $\xi|_\Lambda = \eta$. Then,

\[ \sigma_{L\setminus \Lambda,\xi_\Lambda}(\eta) = \alpha_{L\setminus \Lambda} \sigma(\eta) \mu_{\Lambda,\eta}(\xi_\Lambda) = \sigma_{L\setminus \Lambda}(\eta) \mu_{\Lambda,\eta}(\xi_\Lambda). \]  

(4.1)

This is justified, since $\sigma$ is a Gibbs state:

\[ \alpha_\Lambda \sigma = \sum_{\eta \in \Omega_{L\setminus \Lambda}} \alpha_{L\setminus \Lambda} \sigma(\eta) \mu_{\Lambda,\eta} \]  

(4.2)

(4.1) implies

\[ \sigma_{L\setminus \Lambda,\xi_\Lambda}(\eta)/\mu_{\Lambda,\eta}(\xi_\Lambda) = \sigma_{L\setminus \Lambda,\xi_\Lambda}(\eta)/\mu_{\Lambda,\eta}(\xi_\Lambda) \]  

(4.3)

That is,

\[ \sigma_{L\setminus \Lambda,\xi_\Lambda}(\eta) \mu_{\Lambda,\eta}(\xi_\Lambda) = \sigma_{L\setminus \Lambda,\xi_\Lambda}(\eta) \mu_{\Lambda,\eta}(\xi_\Lambda), \]  

(4.4)

\[ \text{According to this definition, ‘any’ average of ‘local canonical distribution with any external configuration is allowed, so the obtained Gibbs measure can be a superposition of different states.} \]
or
\[ e^{-U_\Lambda(\zeta_\Lambda)-B(\zeta_\Lambda,\eta)}\sigma_{L\setminus\Lambda,\xi_\Lambda}(\eta) = e^{-U_\Lambda(\xi_\Lambda)-B(\xi_\Lambda,\eta)}\sigma_{L\setminus\Lambda,\xi_\Lambda}(\eta). \] (4.5)

Let \( \sigma \in K_\Phi \) be a measure that is not extremal. This implies that there is a non-constant (essentially bounded) positive function \( F \) on \( \Omega \) such that \( F\sigma \) is proportional to a Gibbs state. If we combine this with (4.5), we get
\[ e^{-U_\Lambda(\zeta_\Lambda)-B(\zeta_\Lambda,\eta)}F(\xi_\Lambda \vee \eta)\sigma_{L\setminus\Lambda,\xi_\Lambda}(\eta) = e^{-U_\Lambda(\xi_\Lambda)-B(\xi_\Lambda,\eta)}F(\xi_\Lambda \vee \eta)\sigma_{L\setminus\Lambda,\xi_\Lambda}(\eta) \] (4.6)
for all finite \( \Lambda \subset L \) and all \( \xi_\Lambda,\zeta_\Lambda \in \Omega_\Lambda \). Since \( \sigma \) is a Gibbs state, the above equality without \( F \) also holds. Thus,
\[ F(\xi_\Lambda \vee \eta) = F(\zeta_\Lambda \vee \eta). \] (4.7)
That is, \( F \) does not depend on the configuration on any finite \( \Lambda \). Therefore, \( F \) is in \( B_\sigma \). Therefore, non-extremality implies that the algebra at infinity is not trivial (not constant).

5 Cluster property

This was just as discussed in the class.