Lecture 3. ABC of renormalization group theory

Key concepts: Stückelberg-Petermann renormalization group (RG), Wilson-Kadanoff RG, RG constant, RG equation, anomalous dimension, scaling transformation, Kadanoff transformation, RG transformation.

In the preceding lecture I have discussed that there are two ways to extract the phenomenological-theoretical structure from a set of phenomena or systems under study. One way is to shake the microscopic details and watch which macroscopic observable parts are violently shaken. Let us remove them as ‘materials constants.’ The second way is to look the systems from increasing distances with fixed eyesight (fixed resolving power of observation). The features that become increasingly dominant should be the universal features we observe at the macroscopic scales. In this case, microscopic details are subsumed in various numerical factors to fit the asymptotic relations to actual system data. The first strategy is due originally to Stückelberg and Petermann, and the second due originally to Kadanoff and Wilson.

To outline these two ideas, let us consider a prototypical example: the von Koch curve constructed by the recursive procedure sketched in Figure 3.1. If we repeat this procedure ad infinitum, we obtain a nowhere differentiable but continuous self-similar curve (a fractal curve). We are interested in the total length $L$ of the curve when its ‘diameter’ is $L_0$ and the bond length is $\ell$.

The construction procedure in Fig 3.1 implies

$$\ell = L_0 \to L_0/3 \to \cdots \to L_0/3^n \to \cdots$$  \hspace{1cm} (3.1)

$$L = L_0 \to 4L_0/3 \to \cdots \to (4/3)^nL_0 \to \cdots.$$  \hspace{1cm} (3.2)

Since $n = \log(L_0/\ell)/\log 3$,

$$L = L_0(4/3)^{\log(L_0/\ell)/\log 3} = L_0(L_0/\ell)^{\log 4/\log 3 - 1},$$  \hspace{1cm} (3.3)

that is,

$$L = L_0^{\log 4/\log 3} \ell^{1-\log 4/\log 3}.$$  \hspace{1cm} (3.4)
This has the general structure of the phenomenological description (1.1). Let us collect many von Koch curves with different microscopic details (i.e., $\ell$).

The length of the curve $L$ is proportional to $L_0^{\log 4/\log 3}$. This is the universal feature independent of the microscopic details. The proportionality constant $\ell^{1-\log 4/\log 3}$ is micro-detail sensitive (notice that it diverges in the $\ell \to 0$ limit) and is the materials constant in our example.

Before going into the main topic let us study the example dimensionally analytically. We have $L$, $L_0$, and $\ell$, so there are two dimensionless ratios $L/L_0$ and $L_0/\ell$. Therefore, dimensional analysis tells us that

$$L = L_0 f(L_0/\ell).$$  \hspace{1cm} (3.5)

The classical wisdom mentioned in Lecture 1 does not work: $\ell \to 0$ is not well-defined. The result (3.4) has the following structure

$$L = L_0(L_0/\ell)^{\alpha-1} \propto L_0^{\alpha},$$  \hspace{1cm} (3.6)

where $\alpha - 1$ is called the anomalous dimension, telling us as if the dimension of $L$ is different from that of $L_0$.

Now, I explain the Stückelberg-Petermann renormalization group (field-theoretical RG) approach. First, we ask what we can really observe. Definitely, we can observe $L_0$. Can we observe $L$, the true length? If $\ell$ is smaller than the smallest length $\lambda$ we can discern (the resolution), it is impossible to measure $L$. Therefore, we introduce the length $\tilde{L}$ we measure with the aid of a stick of length $\lambda$. Although the true length is not observable, we expect that $L$ and $\tilde{L}$ are proportional:

$$\tilde{L} = ZL.$$  \hspace{1cm} (3.7)

This is, so to speak, the manifestation of our belief in reality. The quantities we deal with are $L_0$, $\tilde{L}$ and $\lambda$, so dimensional analysis tells us the following functional relation

$$\tilde{L} = \lambda f(L_0/\lambda),$$  \hspace{1cm} (3.8)

where $f$ is determined by a more detailed consideration.

Our strategy is to find the divergence in the $\ell \to 0$ limit. Then, we try to absorb the divergence into the quantity that may be interpreted as an adjustable parameter when we wish to fit our result to the actual result. Since $\ell \to \ell/3$ makes the total length $4/3$ times as large as the curve before, we know

$$L \sim \ell^{1-\log 4/\log 3}. \hspace{1cm} (3.9)$$

We cannot know $L$, so the relation between $\tilde{L}$ and $L$ must be an adjustable parameter. That is, the above divergence must be absorbed into $Z$. Thus,

$$Z \propto \ell^{\log 4/\log 3-1}. \hspace{1cm} (3.10)$$

\footnote{For us this is trivial, because we know everything. Therefore, you may say our strategy is useless, because we need detailed knowledge. However, in practice, divergences (singularities) are marked and are relatively easy to identify. Thus, the present approach has a merit (if you are skeptical, go to Lecture 4 or Lecture 13).}
Note that $Z$ must be dimensionless, so it can be written as a function of dimensionless quantities. Furthermore, it is something like a materials constant, so it should depend on the local properties of the system. Therefore, we may fix

$$Z = \left(\frac{\ell}{\lambda}\right)^{\log 4/\log 3 - 1}.$$  

(3.11)

The numerical factor in the proportionality relation (3.10) may be chosen to be unity: this corresponds to the choice of a particular length unit. $Z$ is called the renormalization group constant.

Now, we demand that World external to us exists independent of how we observe it, so $L$ should not depend on $\lambda$. This implies that

$$\frac{\partial L}{\partial \lambda} = 0,$$  

(3.12)

where we keep $\ell$ and $L_0$ untouched; the quantities specifying the system under study are kept constant. This equation is called the renormalization group equation. It is often convenient to rewrite this as

$$\lambda \frac{\partial \log L}{\partial \lambda} = 0.$$  

(3.13)

(3.7) and (3.8) imply

$$L = Z^{-1} \lambda f(L_0/\lambda).$$  

(3.14)

Putting this into (3.13), we obtain the following renormalization group equation for $f$

$$1 - \alpha - \frac{f(x)}{f(x)} x = 0,$$  

(3.15)

where

$$\alpha = \frac{\partial \log Z}{\partial \log \lambda} \to 1 - \frac{\log 4}{\log 3} \quad (\text{as } \ell \to 0).$$  

(3.16)

We are interested in the limit of small $\ell$, so we may identify $\alpha$ with the $\ell \to 0$ limit value given in (3.16). Integrating this equation, we obtain

$$f(x) = A x^{1-\alpha} = A x^{\log 4/\log 3},$$  

(3.17)

where $A$ is an integration constant. Thus, (3.8) tells us that

$$\tilde{L} \propto \lambda \left(\frac{L_0}{\lambda}\right)^{\log 4/\log 3} = C L_0^{\log 4/\log 3},$$  

(3.18)

where $C$ is an adjustable parameter (‘materials constant’). We have extracted the phenomenological structure equivalent to (3.4).

The von Koch curve problem may be regarded as a prototype of the sea shore line length problem discussed in Mandelbrot’s famous book, *Fractals*. $\lambda$
is the yardstick we measure the shoreline. The shoreline is statistically roughly self-similar. The shoreline teaches us an important lesson. If we magnify it extensively, we would see waves breaking on the shore, so the shoreline fluctuates violently (if definable at all). It is clear that with more magnification, the concept of shoreline becomes meaningless. That is, the meaning of ‘existence’ (or ‘microscopic reality’) can be very delicate.

You might ask, “Why can’t we ask an infinitesimal bug (or a diffusing particle) to walk along the curve?” We would be able to time how long it takes for him to complete his trip, and infer $L$. Perhaps. However, notice that $L$, which is now equal to $\tilde{L}$, is still proportional to $L_0^{\log 4/\log 3}$. What is the lesson to learn?

In the above construction of the von Koch curve, the replacement of each bond with the bent bond is performed uniformly throughout the curve. We can introduce a stochastic von Koch curve in which the replacement is done probabilistically with probability $p$ (Fig 3.2). What result will you obtain?

![Fig 3.2 The stochastic von Koch curve](image)

Now, this probability $p$ need not be the same at every construction step in Fig 3.2. Perhaps $p$ oscillates step by step. Then, $\alpha$ in (3.16) does not have any limit. What can we do in this case? Notice that the RG equation is still useful if $p$ does not rapidly change from step to step.

Let us continue the study of the von Koch curve. Now, we use the Wilson-Kadanoff RG. We define the scaling transformation $S$ and the coarse-graining procedure (Kadanoff transformation) $K$ as in Fig 3.3. We have

$$S(L_0) = L_0/3, \ S(L) = L/3, \ S(\ell) = \ell/3, \quad (3.19)$$
$$K(L_0) = L_0, \ K(L) = 3L/4, \ K(\ell) = 3\ell. \quad (3.20)$$

We define the renormalization group transformation $R$ as $R = SK$. We have

$$R(L_0) = L_0/3, \ R(L) = L/4, \ R(\ell) = \ell. \quad (3.21)$$
Notice, that this transformation is chosen to keep the ‘microscopic feature’ intact. Since
\[ L(n) \equiv R^n(L) = (1/4)^n L, \tag{3.22} \]
and
\[ L_0(n) \equiv R^n(L_0) = (1/3)^n L_0, \tag{3.23} \]
we obtain
\[ L = \left( L_0 / L_0(n) \right)^{\log 4 / \log 3} L(n). \tag{3.24} \]
Choose \( n \) so that \( L_0(n) \) and \( L(n) \) are of order unity, and we conclude \( L \propto L_0^{\log 4 / \log 3} \).

The map \( R \) is called renormalization group transformation, because \( RR = R^2 \) makes sense, this multiplication is associative, and \( R^0 = 1 \) (not applying \( R \) is equivalent to doing nothing), so \( \{ R^n \} \) makes a semigroup with a unit (i.e., monoid).\textsuperscript{14}

You must know the system well to invent a useful \( R \) (it is an art). According to my memory Leo Kadanoff once said something like, “if we know the answer, we can always invent an RG to obtain it.”

I have outlined two representative RG approaches. If you understand their spirit, you have understood RG. The rest is a ‘mere’ technicality.

Anderson writes: “… I believe that the renormalization group has the potential of providing an even more complete conceptual unification of the science of complex systems than has yet been realized, and that the common view of it as a technique useful only in phase transition theory and field theory is a great pity.”\textsuperscript{15}

\textsuperscript{14}The inverse \( R^{-1} \) may not be defined in general, so it is not a true group.
\textsuperscript{15}P W Anderson, Basic Notions of Condensed Matter Physics (Benjamin-Cummings, 1984) p5.