decrease is discounted as
\[ \Delta S = k_B (1 - 1/2) \log N. \] (30.16)

Therefore,
\[ 2\mu^2 J \geq \frac{1}{2} k_B T \] (30.17)
is the condition for spontaneous ordering of spins. However, the nature of the phase transition is delicate.

We have already seen a naive argument against spontaneous ordering (Lecture 23). According to the transverse fluctuations the correlation function diverges (logarithmically). Therefore, spins cannot stably point in a fixed direction. Now, let us state the Mermin-Wagner theorem:190 The Heisenberg model has no spontaneous magnetization if \( d \leq 2 \).

More precisely, let
\[ \mathcal{H}(V) = -J \sum_{\langle i,j \rangle} \mathbf{s}(i) \cdot \mathbf{s}(j) - h \sum_j s_x(j). \] (30.18)

Let \( a(h) \) be the free energy per spin in the thermodynamic limit for this model. Then,
\[ \lim_{h \to 0} \frac{da(h)}{dh} = 0. \] (30.19)

The starting point of the proof is the following Bogoliubov inequality:
\[ \frac{1}{2} \beta \langle AA^\dagger + A^\dagger A \rangle \cdot \langle [C, \mathcal{H}], C^\dagger \rangle \geq \langle [C, A] \rangle^2. \] (30.20)
(A proof is given in Appendix.) Now, set
\[ C = s_z(k), \quad A = s_y(-k). \] (30.21)

We must compute each term in the inequality explicitly.
\[ [C, A] = [s_z(k), s_y(-k)], \] (30.22)
\[ = \sum_r \sum_{r'} [s_z(r), s_y(r')] e^{i\mathbf{k} \cdot (r - r')}. \] (30.23)

Here, notice that the spins at different positions commute, so
\[ [C, A] = \sum_r (-i)s_x(r). \] (30.24)

Therefore, this average is $-imV$ (remember that the external field points in the $x$-direction), so

$$|\langle [C, A] \rangle|^2 = (Vm)^2. \quad (30.25)$$

Since $(s_y(-k))^\dagger = s_y(k)$, we have

$$\langle AA^\dagger + A^\dagger A \rangle = 2\langle s_y(-k)s_y(k) \rangle = 2|s_y(k)|^2. \quad (30.26)$$

The most complicated term is left to your enjoyment:

$$[[C, H], C^\dagger] = hs_x(0) + 2J \sum_{\langle i, j \rangle} [1 - \cos k \cdot (r_i - r_j)] [s_x(i)s_x(j) + s_y(i)s_y(j)], \quad (30.27)$$

$$\leq V \left( |hm| + Js^2|k|^2 \right), \quad (30.28)$$

because $1 - \cos x = 2\sin^2(x/2) \leq x^2/2$. Also note that the number of nearest neighbor pairs = number of lattice points = $V$. Here, $s$ is the maximum value of the spin ($z$-) component.

Combining all the results, the Bogoliubov inequality reads

$$V^2m^2 \leq V \left( |hm| + Js^2|k|^2 \right) \beta|s_y(k)|^2. \quad (30.29)$$

That is,

$$Vm^2 \left( |hm| + Js^2|k|^2 \right)^{-1} \leq \beta|s_y(k)|^2. \quad (30.30)$$

Now, we sum the both sides with respect to $k$

$$m^2V \sum_k \left( |hm| + Js^2|k|^2 \right)^{-1} \leq \beta \sum_k |s_y(k)|^2. \quad (30.31)$$

Honestly compute

$$\sum_k |s_y(k)|^2 = \sum_k \sum_{r, r'} s_y(r)s_y(r')e^{ik \cdot (r-r')}, \quad (30.32)$$

$$= \sum_{r, r'} s_y(r)s_y(r')V\delta_{r, r'}, \quad (30.33)$$

$$= V \sum_r s_y(r)^2 \leq V^2s^2. \quad (30.34)$$

Therefore, (30.31) reads

$$m^2V^{-1} \sum_k \left( |hm| + Js^2|k|^2 \right)^{-1} \leq \beta s^2. \quad (30.35)$$

Now, the sum may be replaced with the integral.\textsuperscript{191} I leave a better proof based on the evaluation of the integral to you, and proceed in a crude fashion. If $h = 0$, then the integral diverges. Since $s$ is bounded, we must conclude

\textsuperscript{191}$V^{-1} \sum_k \to (2\pi)^{-d} \int d^dk.$
that $m$ must be zero in the $h \to 0$ limit. That is, there is no spontaneous magnetization.

The above argument may be adapted to the XY-model.

Appendix. Bogoliubov inequality\textsuperscript{192}

Define a positive semidefinite scalar product in the space of operators defined on a finite dimensional Hilbert space as

$$
(A, B) = \sum_{\phi, \psi} \langle \psi | A^\dagger | \phi \rangle \langle \phi | B | \psi \rangle \frac{W_\phi - W_\psi}{E_\psi - E_\phi},
$$

(30.36)

where $\{|\phi\rangle\}$ and $\{|\psi\rangle\}$ are orthonormal bases consisting of the eigenkets of $H$,

$$
\mathcal{H}|\phi\rangle = E_\phi |\phi\rangle, \quad W_\phi = Z^{-1} e^{-\beta E_\phi},
$$

(30.37)

$$
Z = Tr e^{-\beta \mathcal{H}}, \quad 0/0 \text{ is interpreted as } 0.
$$

First, note that if $y > x$

$$
e^{-x} - e^{-y} \leq \frac{e^{-(x+y)/2}(e^{(y-x)/2} - e^{-(y-x)/2})}{e^{-x+y/2}(e^{(y-x)/2} + e^{-(y-x)/2})} = \tanh \left( \frac{y - x}{2} \right) < \frac{y - x}{2},
$$

(30.38)

that is,

$$
\frac{e^{-x} - e^{-y}}{y - x} < \frac{1}{2} \left( e^{-x} + e^{-y} \right).
$$

(30.39)

This implies

$$
\frac{W_\phi - W_\psi}{E_\psi - E_\phi} < \frac{1}{2} \beta (W_\phi + W_\psi).
$$

(30.40)

Therefore, with the aid of this majorant (Here, $W$ is the density matrix)

$$
(A, A) < \frac{1}{2} \beta \sum_{\phi, \psi} \langle \psi | A^\dagger | \phi \rangle \langle \phi | A | \psi \rangle (W_\phi + W_\psi),
$$

(30.41)

$$
= \frac{1}{2} \beta \sum_{\phi, \psi} \left( \langle \psi | A^\dagger | \phi \rangle W_\phi \langle \phi | A | \psi \rangle + \langle \phi | A | \psi \rangle W_\psi \langle \psi | A^\dagger | \phi \rangle \right),
$$

(30.42)

$$
= \frac{1}{2} \beta (Tr A^\dagger WA + Tr AW A^\dagger) = \frac{1}{2} \beta \left( \langle AA^\dagger \rangle + \langle A^\dagger A \rangle \right).
$$

(30.43)

\textsuperscript{192} Ruelle SM p130-131.
Next, let us compute the scalar product for $B = [C^\dagger, \mathcal{H}]$

\[
(A, [C^\dagger, \mathcal{H}]) = \sum_{\phi, \psi} \langle \psi | A^\dagger | \phi \rangle \langle \phi | (C^\dagger \mathcal{H} - \mathcal{H} C^\dagger) | \psi \rangle \frac{W_\phi - W_\psi}{E_\psi - E_\phi},
\]

\[
= \sum_{\phi, \psi} \langle \psi | A^\dagger | \phi \rangle \left( \langle \phi | C^\dagger | \psi \rangle E_\psi - E_\phi \langle \phi | C^\dagger | \psi \rangle \right) \frac{W_\phi - W_\psi}{E_\psi - E_\phi},
\]

\[
= \sum_{\phi, \psi} \langle \psi | A^\dagger | \phi \rangle \langle \phi | C^\dagger | \psi \rangle (W_\phi - W_\psi),
\]

\[
= \langle [C^\dagger, A^\dagger] \rangle.
\]

This also implies

\[
(B, B) = \langle [C^\dagger, [\mathcal{H}, C]] \rangle.
\]

Putting (30.43), (30.47), and (30.48) into (30.36) gives the desired inequality.