Review

Strong law of large numbers
For almost all (i.e., except for events with probability measure 0) runs

\[ \frac{1}{N} \sum_{n=1}^{N} X_n \to \langle X_1 \rangle. \]

We can observe probabilities:

\[ \frac{1}{N} \sum_{n=1}^{N} \chi_A(X_n) \to P(A). \]

Relative frequencies can be used to get objective Ps.

Thermodynamic and transport phenomena are deterministic.

Kolmogorov’s 01 law
Maxwell’s distribution

5.1: Probability density distribution:
\[ f(x) = \frac{P(dx)}{dx}. \]

Radon-Nikodym derivative:
If \( \mu(A) = 0 \) always implies \( \nu(A) = 0 \)
\[ f \equiv \frac{d\nu}{d\mu} \]
is well defined. This is called the density of \( \nu \) wrt \( \mu \).

5.2 Maxwell distribution via Cauchy’s functional equation:
Normalized result:
\[ f(\mathbf{v}) = \left( \frac{m}{2\pi k_BT} \right)^{3/2} e^{-m\mathbf{v}^2/2k_BT} \]

(1) How can we empirically verify Maxwell’s distribution?
You always need something not in equilibrium.

(2) Quantum mechanics?
Some technical aspects

5.3-5 How to compute Gaussian integrals

5.7 Boltzmann factor $e^{-\beta U}$. 
We know

\[ P(A) = \langle \chi_A(\omega) \rangle. \]

6.1 What is the counterpart for a density?

\[
\begin{align*}
f(y) &= \frac{P(d\tau(y))}{d\tau(y)} = \frac{\langle \chi_d\tau(y)(x) \rangle}{d\tau(y)} = \left\langle \frac{\chi_{dx}(y)(x)}{d\tau(y)} \right\rangle \\
&= \delta(x - y) d\tau(y) = \left\langle \delta(x - y) \right\rangle = \int_\Omega dP(x) \delta(x - y).
\end{align*}
\]

\(\delta\)-function! 6.2, see Fig. 6.1.

Techniques 6.7-9
6.10 How about the density distribution of $Z = G(X)$?

e.g.,

By definition (knowing $P(X)$) $(d\tau(z) \equiv dz)$

$$f(z) = \frac{P(\{x \mid G(x) \in dz\})}{dz} = \langle \delta(z - G(x)) \rangle.$$ 

Why?

$$\frac{P(\{x \mid G(x) \in dz\})}{dz} = \left\langle \frac{\chi_{\{y \mid G(y) \in dz\}}(x)}{dz} \right\rangle;$$

Notice that (intuitively, $1/dx = \delta(x)$)

$$\chi_{\{y \mid G(y) \in dz\}} = \delta(z - G(y))dz$$
Within this theory, you can do whatever you wish to do formally!

14 Generalized Function

The $\delta$-function is not an ordinary function and is meaningful only inside the integral. The theory of distribution in the sense of Sobolev and Schwartz rationalizes such objects like the $\delta$-function. Rudiments of the theory are outlined from the practitioner's point of view. Calculation of Green's functions may be facilitated by the theory of generalized functions which justifies apparent abuses of classical analysis.

**Key words:** generalized function, distribution, test function, Schwartz class, regular distribution, convolution, $\delta$-function, differentiation of $\delta$-function, Heaviside step function, Cauchy principal part.

We get 'weak results,’ but practically in physics they are enough.
Let us practice

6.1
Q6.3
Mean Free path