Away from $T_c$, what happens? 
$\xi$ becomes smaller $\approx$ observing from distance

Watch (Dr Ashton again) 
https://www.youtube.com/watch?v=MxRddFrEnPc

**Kadanoff construction**

At $T_c$ the system is invariant under ‘shrinking and coarse-graining.’

We will go through 33.9, 33.10 slowly.

**Kadanoff ansatz:**

Let $\ell$ be the shrinking rate ($> 1$)

\[ \tau = (T - T_c)/T_c \rightarrow \tau \ell^{y_1} \]

\[ h \rightarrow h\ell^{y_2}. \]

$y_1 > 0, y_2 > 0$. 
To perfect up phase
to perfect down phase
to high temperature limit
A
A'
B
C
V
T
Solid
triple point
CP
B
C
V
T
Gas
supercritical fluid
Solid
Liquid
Magnet
$H$
to perfect up phase
to high temperature limit
to perfect down phase
A
A'
B
C
V
T
actual
$n > 1$
$n = 0$
renormalization transformation
$n >> 1$
fixed point $0$
$n >> 1$
$n > 1$
$n = 0$
unstable mfd of $H^*$

$T_c$

for magnet A

$T_c$

for magnet B

critical surface

= stable mfd of $H^*$

where linearization is OK
Renormalization Group: Basic ideas

from Kadanoff
(1) Apply $\mathcal{R}$ many times and study the properties around the fixed point.

Another approach
(2) Pursue invariants under microscopic perturbations
\[
\frac{L}{W} = f \left( \frac{W}{\ell} \right), \quad \text{(L24.1)}
\]

\( \ell = W/3^n \)

the total length of the curve is \( L = \left( \frac{4}{3} \right)^n W \). Since

\[
n = \log_3 \left( \frac{W}{\ell} \right), \quad \text{(L24.2)}
\]

we obtain

\[
L = W^{\log 4/\log 3} \ell^{1-\log 4/\log 3}. \quad \text{(L24.3)}
\]

That is, \( f(x) = x^{\log 4/\log 3 - 1} \)
Wilson-Kadanoff approach

\[ S(\ell) = \ell/3 \] (indeed, \( S(\mathcal{K}(\ell)) = \ell \)). Obviously, \( S(L) = L/3 \) and \( S(W) = W/3 \).

\[ \mathcal{R} \equiv \mathcal{S}\mathcal{K}; \text{ renormalization group transformation} \]

Since \( \mathcal{R}(L) = L/4 \), we obtain

\[ \mathcal{R}^n(L) \equiv L(n) = L/4^n. \tag{L24.4} \]

Here, \( \mathcal{R}(W) = W/3 \).

\( n \): dilation parameter to count \# of applied \( \mathcal{R} \)

\[ W(n) = \mathcal{R}^n(W) = W/3^n \Rightarrow n = \ln\{W/W(n)\} / \ln 3. \]

Then, (??) may be rewritten as

\[ L = 4^n L(n) = \left( \frac{W}{W(n)} \right)^{\ln 4/\ln 3} L(n). \tag{L24.5} \]

One approach: reduce the problem to a simpler one.
Another approach: Find the fixed point of $\mathcal{R}$ and study its behavior in its nbh.

$1/L = 0$ is the fixed point. $\mathcal{R}$ to be $W \to W/2$. Then,

$$(1/L) \to 2^{\log_4/\log_3}(1/L)$$

$\log_2$ eigenvalue = exponent.

**Lecture 24 RG**
Here, let us see the simpler one (see 33.15 by yourself)

1D decimation

\[ Z = \sum_{s,\sigma} \cdots e^{K(s_{-1}\sigma_0 + \sigma_0 s_1)} \cdots, \]

Even lattice spins: \( \sigma \).
Sum over all \( \sigma \) states and make a decimated chain

\[ Z = \sum_{s} \cdots e^{A+K's_{-1}s_1} \cdots. \]

Find \( K' \) in terms of \( K \):
If \( s_{-1} + s_1 = 0 \): \( 2 = e^{A-K'} \);
Otherwise, \( 2 \cosh(2K) = e^{A+K'} \),
\[ \Rightarrow e^{2K'} = \cosh(2K). \]

\[ K' = \frac{1}{2} \log \cosh 2K. \]

\[ H = \sum_{i\in\mathbb{Z}} K s_i s_{i+1} \rightarrow H' = \sum_{i\in\mathbb{Z}/2} K' s_i s_{i+2}. \]

\( K \rightarrow K' \rightarrow \cdots \rightarrow 0 \) quickly \( \Rightarrow \) no phase transition for \( T > 0 \).
Stückelberg-Petermann approach:

“To pursue invariants under changing microscopic details.”

Key point: The true length $L$ is not observable, but “condition of realism”: The result of observation at scale $\lambda$ is still $\propto L$:

$$\tilde{L} = Z(\ell/\lambda)L.$$

$L$: renormalization constant

How badly does $L$ behave as a function of ‘cutoff’ $\ell$?

$L$ in the $\ell \to 0$ limit should behave as $(4/3)^{-\log_3 \ell} = \ell^{1-\log 4/\log 3}$.

$\ell$ is shrunk by factor 3 to $\ell/3 \Rightarrow n = -\log_3 \ell$

total length of the curve is multiplied by 4/3.

$Z$ should be selected to remove this divergence $\ell^{1-\log 4/\log 3}$ (i.e., $ZL$ does not have any divergence): $Z(\lambda/\ell) \propto (\lambda/\ell)^{1-\log 4/\log 3}$

$$\lambda \frac{\partial L}{\partial \lambda} = 0. \quad \text{(L24.7)}$$

$f$ here is also a well-behaved function. From (??) and this, we have

$$L = Z^{-1} \lambda f \left( \frac{W}{\lambda} \right). \quad \text{(L24.8)}$$

Introducing this into (??), we obtain (it is wise to compute the log-

arithm derivative $\partial \log L/\partial \log \lambda$)

$$f(x) - \alpha f(x) - xf'(x) = 0, \quad \text{(L24.9)}$$

where

$$\alpha \equiv \partial \log Z/\partial \log \lambda = 1 - \log 4/\log 3. \quad \text{(L24.10)}$$

Solving (??), we obtain

$$f(x) \propto x^{1-\alpha} \Rightarrow \tilde{L} \propto W^{1-\alpha} \lambda^\alpha \propto W^{\log 4/\log 3}. \quad \text{(L24.11)}$$
SAW or polymer in good solvent

Perturbative calculation:

qualitative change ⇒ series must diverge

The Boltzmann factor for a conformation:

$$\exp \left[ -\frac{1}{2} \int_0^{N_0} d\tau \int_0^{N_0} d\sigma \, v(r(\tau) - r(\sigma)) \right]$$  \hspace{1cm} (L24.12)

$$G_B(\mathbf{R}, N_0) = \langle \exp \left[ -\frac{1}{2} \int_0^{N_0} d\tau \int_0^{N_0} d\sigma \, v(r(\tau) - r(\sigma)) \right] \rangle_{0, R} G_0(\mathbf{R}, N_0),$$  \hspace{1cm} (L24.13)

where ‘without interactions’

$$G_0(\mathbf{R}, N_0) = (2\pi)^{-d/2} e^{-R^2/2N_0}.$$  \hspace{1cm} (L24.14)

$$v(r) = v_0 \delta(r) \text{ [Edwards model]}$$

Microscopic details:

monomer size, number of monomers $N_0$, interaction strength $v_0$. 

10
Perturbation:

\[
\frac{G_B(R, N_0)}{G_0(R, N_0)} = 1 - \frac{1}{2} \left\langle 0 \int_0^{N_0} d\tau \int_0^{N_0} d\sigma \, v(r(\tau) - r(\sigma)) \right\rangle_{0, R} + \cdots.
\]  

(L24.15)

\[
G_B(R, N_0) = G_0(R, N_0) - v_0 I(R, N_0) + O[v_0^2], \quad (L24.16)
\]

where we may write

\[
I(R, N_0) = \int_0^{N_0} d\tau \int_0^{\tau} d\sigma \int d^3 R \, G_0(R - R, N_0 - \tau) G_0(0, \tau - \sigma) G_0(R, \sigma) = \int_0^{N_0} ds (N_0 - s) G_0(R, N_0 - s) G_0(0, s).
\]  

(L24.17)
Dimensional technique

Perform dimensional analysis:

Compare the unperturbed term and the first order term in \( ??? \).

\[
[G] = [v_0][G]^3[d^d\mathcal{R}][d\tau][d\sigma] \Rightarrow [v_0] = L^{-\epsilon}, \text{ where } \epsilon = 4 - d
\]

Perturbation in \((4 - \epsilon)-space\)

\[
I = (2\pi)^{-d}N_0^{-2}e^{-\alpha}G(\alpha) \quad \text{(L24.18)}
\]

\[
G(\alpha) = \int_\ell/N_0^\infty dt \ (1 + t)t^{-2}e^{-\alpha t} \ . \quad \text{(L24.19)}
\]

with \( t = s/(N_0 - s) \).

\[
G(\alpha) = \int_\ell/N_0^\infty dt \ (1 + t)(- \log t)^{\prime\prime}e^{-\alpha t} \quad \text{(L24.20)}
\]

\[
= - \left( \frac{1}{t} + 1 \right) e^{-\alpha t} \bigg|_{\ell/N_0}^\infty + [(1 - \alpha) - \alpha t] e^{\alpha t} \log t \bigg|_{\ell/N_0}^\infty + \text{regular terms,} \quad \text{(L24.21)}
\]

we need not retain all the terms not containing \( \alpha \propto R^2 \) (keep convenient ones, if any). Therefore, we obtain

\[
G_B(R, N_0) = G_0(R, N_0) \left\{ 1 + v_0(2\pi)^{-2}(1 - \alpha) \log(\ell/N_0) + \text{regular terms} \right\} . \quad \text{(L24.22)}
\]
Renormalization

\[ N = Z_N N_0, \quad v = Z_v v_0. \quad (L24.23) \]

We can absorb the singularities also into the multiplicative factor in front of the density distribution function of \( R \): \( G_B \) (\( B \) implies ‘bare’)

\[ G(R, N) = Z_G G_B(R, N_0). \quad (L24.24) \]

Perturbation!

\[ Z_N = 1 + Av + O[v^2], \quad (L24.25) \]
\[ Z_v = 1 + Bv + O[v^2], \quad (L24.26) \]
\[ Z_G = 1 + Cv + O[v^2]. \quad (L24.27) \]

Introducing these into \( G \), we have (we ignore regular terms)

\[ G(R, N) = Z_G[G_0(R, Z_N^{-1} N)\{1 + Z_v^{-1} v(2\pi)^{-2}(1 - \alpha) \log(\ell/N)\}], \quad (L24.28) \]

Introduce the observation length scale \( \lambda \) and split

\[ \log(\ell/N) = \log(\ell/\lambda) + \log(\lambda/N). \]
\[ G_0(\mathbf{R}, Z_N^{-1}N) = G_0(\mathbf{R}, N)[1 - A \nu(\alpha - 2) + \cdots] \quad (L24.29) \]

and obtain
\[ C = -A = \frac{1}{4\pi^2} \log(\ell/\lambda). \quad (L24.30) \]

Thus, we may write
\[ G(\mathbf{R}, N) = G_0(\mathbf{R}, N)\{1+(2\pi)^{-2}\nu(1-\alpha) \log(\lambda/N)+\text{regular terms}\}. \quad (L24.31) \]

\[ \langle R^2 \rangle = (4 - \epsilon)N + \frac{\nu}{\pi^2}N \log(N/\lambda) + \cdots. \quad (L24.32) \]

Renormalization is not yet complete. We need the next order computation.
What kind of local double interactions may look like a single effective binary interaction in the $\ell \to 0$ limit. These are the four conformations:

\begin{align*}
v = v_0 - 4v_0^2 \int d^4r \int_0^1 dx \int_0^1 dy G_0(r, x) G_0(r, y) + O[v_0^2]. \quad \text{(L24.33)} \\
v = v_0 + v_0^2 \frac{1}{\pi^2} \log \ell + \cdots. \quad \text{(L24.34)}
\end{align*}

Thus, $B = (1/\pi^2) \log(\ell/\lambda)$.

We have obtained

\begin{equation*}
\langle R^2 \rangle = f(v, N). \quad \text{(L24.35)}
\end{equation*}
Renormalization group equation for $\langle R^2 \rangle$.

$$\frac{\partial \langle R^2 \rangle}{\partial \log \lambda} = 0,$$  
(L24.36)

where $N_0$, $v_0$ and $\ell$ are fixed: Reality fixed.

We must find a relation between $\langle R^2 \rangle$ and the observable parameters $N$ and $v$. Dimensional analysis tells us

$$\langle R^2 \rangle = \lambda f(v \lambda^{\epsilon/2}, N/\lambda).$$  
(L24.37)

$$\langle R^2 \rangle = \lambda f(Z_v v_0 \lambda^{\epsilon/2}, Z_N N_0/\lambda).$$  
(L24.38)

Putting this into (??), we obtain

$$f + \frac{\partial f}{\partial x} \beta(x) + y \frac{\partial f}{\partial y} [\gamma(x) - 1] = 0,$$  
(L24.39)

where

$$\beta(x) = \left[ \frac{\partial \log Z_v}{\partial \log \lambda} + \frac{\epsilon}{2} \right] v = \left( -\frac{1}{\pi^2} x + \frac{\epsilon}{2} \right) x,$$  
(L24.40)

$$\gamma(x) = \frac{\partial \log Z_N}{\partial \log \lambda} = \frac{1}{4\pi^2} x.$$  
(L24.41)

The characteristic equations are give by

$$\frac{dx}{\beta(x)} = \frac{dy}{y(\gamma(x) - 1)} = \frac{-df}{f} = \frac{dt}{t},$$  
(L24.42)
\[
\frac{dx}{d\log y} = \beta(x)(\gamma(x) - 1) = x \left( \frac{\epsilon}{2} - \frac{x}{\pi^2} \right) \left( \frac{x}{4\pi^2} - 1 \right)
\]  
(L24.43)

We are interested in large \( y \): \( x \to \epsilon \pi^2/2 \).

\[
\frac{d \log f}{d \log y} = \frac{1}{1 - \gamma^*}.
\]  
(L24.44)

\[
\langle R^2 \rangle \propto f(x^*, y) = y^{1/(1-\gamma^*)} \propto N^{1/(1-\gamma^*)}
\]  
(L24.45)

Therefore,

\[
\nu = \frac{1}{2} \left( 1 + \frac{\epsilon}{8} + \cdots \right).
\]  
(L24.46)

Details:

Forgetful self-avoiding walk \((k\)-th order self-avoiding walk\)

\[
\langle R^2 \rangle = A_k N + B_k
\]  

(L24.47)

In 2-space \(\nu = 3/2\). (Conformal field theory)

In \(d\)-space \((d > 3)\) \(\nu = 1/2\). (Hara-Slade theorem).