Phase transition or not?

(1) Spatial dimensionality
(2) Interaction range
(3) ‘Spin dimension.’ cf Mermin-Wagner Theorem, later.

How to study the phase diagram? 32.9

(1) Phase boundaries = phase transitions
   fluctuation can matter greatly → RG
(2) Bulk properties
   Mean-field effective theories.

Universality:
Dr Ashton’s excellent demos on universality
http://www.youtube.com/watch?v=Kd4UvhUsBAU
\textbf{d-Ising models}

\textbf{1-space:} no transition for $T > 0$
   Fluctuations are too big.

\textbf{2-space:} $\exists$ order-disorder phase transition
   BUT fluctuations are still very large
   (i) No phase coexistence (Higuchi-Aizenman theorem)
      \[ K \text{ is a 1D line segment with two pure states.} \]
   (ii) Critical fluctuations are non-trivial
      \[ \text{Correlation is too strong: ordinary CLT fails.} \]

\[ \text{[demo] use} \]
\[ \text{http://physics.weber.edu/schroeder/software/demos/IsingModel.html} \]
Transversal fluctuations
See the spin ‘field’ from far away:

\[ \mathcal{H} = \frac{1}{2} (\nabla \phi)^2 + V(\phi) \]

Ordering in the \( n \)-th coordinate direction \( \langle \phi \rangle = e_n \).

\[ \delta \phi = \theta + \langle \phi \rangle \]

Think of the Coulomb potential for the transverse fluctuations.

\[ H = \frac{1}{2} (\nabla \theta)^2 + \frac{1}{2} V_{nn} \theta_n^2 + \cdots \]
3-Ising model
Order-disorder phase transition exists ($T_c > 0$).
(i) If $T$ is low enough, two phases coexists. Infinitely many pure states! [Dobrushin’s theorem]
(ii) If not so low, the phase boundary is rough (roughening transition seems to occur around $T = T_c/2$
(iii) Critical fluctuations are non-trivial.

$d(\geq 4)$-Ising models
Order-disorder phase transition exists ($T_c > 0$).
(i) If $T$ is low enough, two phases coexists. Infinitely many pure states!
(ii) $\exists$ roughening transition (?)
(iii) Critical fluctuations are trivial.

Recall $\#$ of communication paths between two spins.
Lee-Yang theory see LeeYang.pdf

For the first time, $\exists$ thermodynamic limit of fluid systems is proved.
Outline: GCE (which is entire)

$$\Xi_V(z) = \sum_{N=0}^{M} z^N Z_N(T, V),$$

The pressure $P_V$ is holomorphic wrt $z$:

$$\frac{P_V}{k_BT} = \frac{1}{V} \log \Xi_V(z).$$

No phase transition.
Number density: if \( \exists \) monotone-increasing, bounded
\[
\frac{\partial}{\partial \log z} \beta P_V = n_V > 0.
\]
uniformly bounded from above.

Proof of Thermodynamic limit
(1) Surface effect asymptotically ignorable
(2) If \( \{ V \} \) is a nested cube sequence, \( \{ (1/V) \log \Xi_V(z) \} \) converges.
(3) For any nested sequence of volumes of any (not too bad) shape, the large volume limit converges.

Essentially, we need (2).
We demonstrate **subadditivity** of $\log \Xi_V$ or

$$
\Xi_{nV} \leq \Xi_V^n \sigma_{nV}
$$

where $\sigma_{nV}$ is the surface contribution $\propto (nV)^{2/3}$ (cf (5.3.6)).

Fekete’s lemma: $\lim \log \Xi_V/V \to \inf_V \log \Xi_V/V$

Thus, $P$ exists.

(1) In $D$, where $\Xi_V$ has no zero for all large $V$, $\lim_{V \to \infty} \beta P_V$ is holomorphic (Vitali).

(2) All the zeros of $e^{\beta P}$ are accumulation points of zeros of $e^{\beta P_V}$ (Hurwitz).

$\Rightarrow$

Singularities of $P$ is the limit of singularities of the finite volume results.
Lee-Yang zero and phase transition

(1) If the zeros of $\Xi$ comes to the real positive axis, there is a phase transition.
(2) If $z$ is small enough, we know $P$ is holomorphic.

cf. Virial equation of state
Lattice gas on cubic lattice see CircleTheorem.pdf

\[ [D]: \text{# of down spins} = \text{# of particles} \]
\[ [U]: \text{# of up spins} = \text{# of empty site} \]
\[ V = [D] + [U], \ M = [U] - [D] = V - 2[D] \]

\[ [UD]: \text{# up-down spin pairs, etc.} \]
\[ \sum_{\langle i,j \rangle} s_i s_j = 3V - 2[UD]. \]

We know
\[ 6[D] = 2[DD] + [UD] \Rightarrow 3V - 2[UD] = 3V - 12[D] + 4[DD] \]

\[-J \sum_{\langle i,j \rangle} s_i s_j - h \sum_i s_i = -J(3V - 12[D] + 4[DD]) - h(V - 2[D])\]
\[\quad = -(3J + h)V + (2h + 12J)[D] - 4J[DD]\]
\[ Z = e^{-\beta V} = \sum [D] e^{\beta (3J+h)V} e^{-\beta (2h+12J)[D]} e^{4\beta J[DD]} \]

Rewrite this as
\[ \Xi = e^{\beta PV} = e^{-\beta V(f+3J+h)} = \sum [D] e^{-\beta (2h+12J)[D]} \left[ \sum e^{4\beta J[DD]} \right] \]

Interpret \( \mu = -(2h + 12J) \rightarrow \text{Lattice gas} \)
The zero points matter:

\[
\Xi = e^{\beta PV} = \sum_{[D]} e^{-\beta(2h+12J)[D]} \left[ \sum e^{4\beta J[DD]} \right]
\]

\[
= \sum_{[D]} e^{-2\beta h[D]} \left[ \sum e^{\beta J(4[DD]-12[D])} \right]
\]

\[
= \sum_{[D]=0}^{V} e^{-2\beta h[D]} \left[ \sum_{\text{with fixed } [D]} e^{-2\beta J[UD]} \right].
\]

Consider this as a degree \( V \) polynomial of \( y = e^{-2\beta h} \).
(Our lattice gas fugacity \( z = ye^{-12\beta J} \).)

\[
\Xi = \sum_{N=0}^{V} y^N Z_N,
\]

\[
Z_N = \sum_{[D]=N} e^{-2\beta J[UD]}
\]
\[ \Xi = \sum_{N=0}^{V} y^N Z_N, \quad Z_N = \sum_{[D]=N} e^{-2\beta J[UD]} \]

Here
\[ Z_N = \sum_{[D]=N} e^{-2\beta J[UD]} = \sum_{|X|=N} \prod_{i \in X, j \notin X} e^{-2\beta J[i,j]}. \]

\(X\): down domain

Therefore,
\[ \Xi = \sum_{X \subseteq L} y^{|X|} \prod_{i \in X, j \notin X} e^{-2\beta J[i,j]}. \]

**Lee-Yang circle theorem**

For \(i, j \in L = \{1, \cdots, V\}\) \(a(i, j) \in [-1, 1]\). Then all the zeros of
\[ P(y) = \sum_{X \subseteq L} y^{|X|} \prod_{i \in X, j \notin X} a(i, j) \]

is on the unit circle.
Hence, the attractive lattice gas on any graph (random lattices included)

(1) the lattice gas has only two phases
(2) the phase transition occurs at $z = e^{-12\beta J}$

However, for the Ising model nothing nontrivial follows since $y = 1$ for $h = 0$; it tells only that with $h \neq 0$ NO phase transition.
Asano’s proof of LY circle theorem
See Circle Theorem.pdf.
Importance of spin dimension

or importance of transverse fluctuations

Why doesn’t Peierl’s type argument work? $1 - \cos \theta \approx \theta^2 / 2$ is the twist energy cost.

$$\int_a^R S_{d-1} \left( \frac{\pi}{R} \right)^2 r^{d-1} dr \approx R^{d-2}$$

This is the energetic cost of flipping a spin with spin dimension $> 1$.

This is hard to rigorize.

2-space is really delicate.

Mermin-Wagner theorem says the magnetization can never be non-zero in 2-space.

Frölich-Simon-Spencer theorem says there is a spontaneous magnetization in 3-space.
Mermin-Wagner theorem see MerminWagner.pdf

\[ \mathcal{H}(V) = -J \sum_{\langle i,j \rangle} \mathbf{s}(i) \cdot \mathbf{s}(j) - h \sum_j s_x(j). \]

Let \( a(h) \) be the ‘free energy’ per spin in the thermodynamic limit for this model. Then,

\[ \lim_{h \to 0} \frac{d a(h)}{dh} = 0. \]

Classic case using ‘classic Bogoliubov + classic angular momenta’

\[ \langle |A|^2 \rangle \geq \frac{k_B T \langle [C, A^*] \rangle^2}{\langle [C, [C^*, H]] \rangle} \]

Cauchy-Schwarz implies

\[ \langle |A|^2 \rangle \langle |B|^2 \rangle \geq |\langle A^* B \rangle|^2. \]

Set \( B = [C, H] \) (\( [\ ] \) is the Poisson Bracket)

\[ \langle A^*[C, H] \rangle = \frac{1}{Z} \int d\Gamma A^*[C, H] e^{-\beta H} = \frac{-k_B T}{Z} \int d\Gamma A^*[C, e^{-\beta H}] \]

\[ = \frac{-k_B T}{Z} \int d\Gamma [A^*, C] e^{-\beta H} = k_B T \langle [C, A^*] \rangle \]

etc.
Correlation functions

\[ C(x - y) = \langle s_x s_y \rangle. \]

Correlation length \( \xi \)

\[ \xi = - \lim_n \frac{\log C(n)}{n}. \]

Is this well-defined?

\( a_n = - \log C(n) \) is subadditive, so Fekete tells us \( a(n)/n \rightarrow \xi \) converges.
Griffiths’ inequality $\langle s^X s^Y \rangle - \langle s^X \rangle \langle s^Y \rangle \geq 0$

$\langle s_0 s_{n+m} \rangle = \langle s_0 s_n s_n s_{n+m} \rangle \geq \langle s_0 s_n \rangle \langle s_0 s_m \rangle$

Thus $a_n = -\log \langle s(0)s(n) \rangle$ is subadditive.

$$\lim a_n/n \geq \inf a_n/n \text{ or } a_n \geq n\xi.$$ That is,

$$C(n) \leq Ae^{-n/\xi}.$$ 

Long-range order $\iff$ spontaneous magnetization.

$\xi$ diverges at $T_c$. 

Let us consider two pure states \( \alpha \) and \( b \).

\[
\langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle = p_\alpha \langle s_i s_j \rangle_\alpha + p_\beta \langle s_i s_j \rangle_\beta
\]

\[
\langle s_i \rangle \langle s_j \rangle = (p_\alpha \langle s_i \rangle_\alpha + p_\beta \langle s_i \rangle_\beta)(p_\alpha \langle s_j \rangle_\alpha + p_\beta \langle s_j \rangle_\beta)
\neq p_\alpha \langle s_i \rangle_\alpha \langle s_j \rangle_\alpha + p_\beta \langle s_i \rangle_\beta \langle s_j \rangle_\beta. \quad (0.0.1)
\]

Therefore, cluster property fails.
What is $C(r)$ at criticality? see correlation.pdf

What happens if $\xi \to \infty$?

$$C(r) \sim \frac{1}{r^{d-2+\eta}}$$

with $\eta \geq 0$. 