Assume that the binary interaction potential decays spatially faster than \( r^{-d} \) in \( \mathfrak{d} \)-space: let us assume \( r^{-(d+\epsilon)} \) for some \( \epsilon > 0 \). Let us divide the \( \mathfrak{d} \)-cube of edge \( L \) into two halves and estimate the interaction energy \( \Delta E \) between them (Fig. 0.0.1).

![Diagram of a \( \mathfrak{d} \)-cube divided into two halves](image)

**Figure 0.0.1:** A \( \mathfrak{d} \)-cube of edge \( L \) is smoothly divided into two halves.

Let us divide the volume into slices of thickness \( dr \), and collect the interaction energies between two slices located at \( r_1 \) and \( r_2 \) along one edge. Let \( \mathbf{x} \) and \( \mathbf{y} \) be the \( \mathfrak{d} - 1 \)-dimensional coordinates in each slice. We also assume that the system is uniform with the number density \( n \). We have

\[
\Delta E = n^2 \int_{L^{d-1}} dx \int_{L^{d-1}} dy \int_{0}^{2L} dr_1 \int_{L+\delta}^{2L} dr_2 \left( \frac{1}{|x - y|^2 + (r_1 - r_2)^2(\delta + \epsilon)/2} \right). \tag{0.0.1}
\]

Here, the contribution from the two slices of thickness \( \delta \) around the center cutting plane is separated. Its contribution must be proportional to \( L^{d-1} \), so we may ignore it relative to the total potential energy of the bulk (\( \propto L^d \)).

\[
\Delta E = \frac{n^2}{2} \int_{L^{d-1}} dx \int_{L^{d-1}} dy \int_{L+\delta}^{L} dr_1 \int_{4\delta^2}^{L^2} dz \left( \frac{1}{|x - y|^2 + z(\delta + \epsilon)/2} \right). \tag{0.0.2}
\]

\[
\Delta E = \frac{n^2}{d - 2 + \epsilon} \int_{L^{d-1}} dx \int_{L^{d-1}} dy \left[ \frac{1}{(|x - y|^2 + 4\delta^2)(\delta - 2 + \epsilon)/2} - \frac{1}{(|x - y|^2 + L^2)(\delta - 2 + \epsilon)/2} \right]. \tag{0.0.3}
\]

The second term does not exceed \( L^{2d - 2 - (d - 2 - \epsilon)} = L^{d - \epsilon} \). The first term term must be smaller than the case assuming that \( x \) can become \( 4L \), so

\[
\int_{L^{d-1}} dx \int_{L^{d-1}} dy \frac{1}{(|x - y|^2 + 4\delta^2)(\delta - 2 + \epsilon)/2} \leq \frac{L^{d-1} S_{d-2}}{L^{d-1} S_{d-2} L^{d-1} S_{d-2}} \int_{0}^{4L} dr \frac{r^{\delta - 2}}{(r^2 + 4\delta^2)(\delta - 2 + \epsilon)/2} \tag{0.0.4}
\]

\[
= L^{d-1} S_{d-2} \int_{0}^{16L^2} dz \frac{z^{(\delta - 3)/2}}{(z + 4\delta^2)(\delta - 2 + \epsilon)/2}. \tag{0.0.5}
\]

\[
\int_{L^{d-1}} dx \int_{L^{d-1}} dy \frac{1}{(|x - y|^2 + z(\delta + \epsilon)/2)} \leq L^{d-1} S_{d-2} \int_{0}^{4L} dr \frac{r^{\delta - 2}}{(r^2 + 4\delta^2)(\delta - 2 + \epsilon)/2} \tag{0.0.6}
\]

\[\text{1}\text{The molecular potential is not necessarily binary, so, precisely speaking, } U \text{ cannot be written as a sum of binary interactions. Here, for simplicity, we assume that we can majorize } |U| \text{ by a sum of binary interactions of the form above.}\]
Here, $S_{\delta-2}$ is the volume of the $\delta - 2$-unit sphere. This integral may be estimated in terms of its primitive function; its value for 0 (the lower bound of the integral) is independent of $L$, so its contribution to $\Delta E$ is bounded by $L^{\delta-1}$. The contribution from the primitive function at the upper bound $16L^2$ is bounded by $L^{\delta-1} \times (L^2)^{(\delta-3)/2-(\delta-2+\epsilon)/2+1} = L^{\delta-\epsilon}$. Thus, $\Delta E$ may be ignored relative to $L^\delta$ in the large volume limit.\footnote{Within the macroscopic thermodynamic framework, we ignore the surface energy. If noted otherwise, the surface energy is ignored in this book.}

Thus, if the interaction potential decays faster than $r^{-\delta}$, then we may assume that the total energy is proportional to the volume of the system.