Yang and Lee’s general theory of grand partition function tells us that the phase transition points may be understood as the crossing points of natural boundaries of the pressure with the real fugacity axis, when the pressure is considered as an analytic function of fugacity $y$.

Now, let us explicitly study an example of fluid: lattice gas. As I mentioned very briefly, up or down spins could be interpreted as particles. Thus, the ferromagnets can be mapped onto lattice gases. Our ferromagnet is the Ising model whose Hamiltonian is given as usual by

$$\mathcal{H} = -J \sum_{\langle i, j \rangle} s_i s_j - h \sum s_i. \quad (27.1)$$

In this lecture I explicitly write the temperature factor $\beta = 1/k_B T$. Let us interpret the down spins in the up spin ocean the particles. In terms of the number of up and down spins, denoted, respectively, as $[U]$, $[D]$, and the number of nearest neighbor pairs, denoted as $[UU]$, $[UD]$, and $[DD]$ the Ising ‘potential’ energy $U$ may be written as

$$U = h([D] - [U]) + 2J[UD], \quad (27.2)$$

where the origin of the energy is chosen to be that of the perfectly ordered phase with $h = 0$ (the vacuum).

Now, let us regard the down spins as particles. The particle particle interactions occur when particles occupy the neighboring lattice points. That is, the number $[DD]$ is the number of interaction pairs. Let us rewrite (27.2) in terms of $[D]$ and $[DD]$. Note that

$$[U] - [D] = M, \quad (27.3)$$
$$[U] + [D] = V, \quad (27.4)$$

where $M$ is the magnetization, and $V$ is the volume or the number of lattice points. If the lattice is the cubic lattice, then counting the number of all the up (down) spins, we obtain

$$2[UU] + [UD] = 6[U], \quad (27.5)$$
$$2[DD] + [UD] = 6[D]. \quad (27.6)$$

Therefore,

$$[U] = V - [D], \quad (27.7)$$
$$[UD] = 6[D] - 2[DD]. \quad (27.8)$$

Incidentally, the number density is $[D]/V$, so the specific volume $v$ (its reciprocal) is given by

$$v = 2/(1 - m), \quad (27.9)$$
The Ising energy now reads
\[ U = h(2[D] - V) + 2J(6[D] - 2[DD]) = -hV + (2h + 12J)[D] - 4J[DD]. \] (27.10)
Let \( f \) be the free energy of the Ising model per site. Then (\( \beta = 1/k_BT \) as usual),
\[ e^{-\beta V f} = \sum \exp(-\beta (-hV + (2h + 12J)[D] - 4J[DD])), \] (27.11)
where the summation is over all the possible down spin configurations. That is,
\[ e^{-\beta (f + h)} = \sum \exp(-\beta ((2h + 12J)[D] - 2J[DD])). \] (27.12)
Therefore, if we set
\[ y = e^{-\beta (2h + 12J)}, \quad p = -(h + f), \] (27.13)
the grand canonical ensemble of the lattice gas and the canonical partition function of the Ising model have been directly related. The translation table reads

<table>
<thead>
<tr>
<th>Ising model</th>
<th>Lattice gas</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of spins</td>
<td>( V )</td>
</tr>
<tr>
<td>number of down spins ( [D] )</td>
<td>number of particles</td>
</tr>
<tr>
<td>( 2/(1 - m) )</td>
<td>specific volume ( v )</td>
</tr>
<tr>
<td>(-f - h)</td>
<td>( p )</td>
</tr>
</tbody>
</table>

Thus, the \( mh \)-diagram and the \( \rho p \)-diagram are mutually translatable as we noted in Lecture 6.

Now, let us study the complex zeros of the grand partition function \( \Xi_V \) of the lattice gas. From the conversion table we know that
\[ e^{-\beta V f} = e^{\beta V h} \Xi_V, \] (27.14)
where \( \Xi_V \) is a polynomial of order \( V \) of \( z = e^{-2\beta h} \)
\[ \Xi_V = \sum_{\ell=0}^{V} z^\ell P_\ell, \] (27.15)
where \( P_\ell \) is the partition function for the Ising model with \( \ell \) down spins.\(^{143}\)

A surprise is that all the zeros of (27.15) is on the unit circle. Therefore, the lattice gas has at most one phase transition point (given by \( y = e^{12J} \) for

\(^{143}\) Why do we not need \( 1/N! \) in (27.15)?
our model). This is true in any dimension, for any lattice, and for any (finite) range of interactions. Even the periodicity of the lattice does not matter.

Let us clearly state what we must prove.

**Theorem 3.** Let the real numbers \( a(\{i, j\}) \in [-1, 1] \) for all \( \{i, j\} \ (i, j \in I = \{1, \ldots, V\}) \). Then, all the zeros of the following polynomial are on the unit circle:

\[
P(z) = \sum_{X \subset I} z^{|X|} \prod_{i \in X, j \notin X} a(\{i, j\}). \tag{27.16}
\]

Here, the terms of order 0 and the highest order are, respectively, 1 and \( z^V \), and \( |X| \) is the number of elements in \( X \) (the cardinality of \( X \)). The summation is over all the subsets of \( I \).

Notice that the summation is over the patterns of down spin islands as used in Lecture 25. \( X \) is a possible set of lattice points with down spins. Therefore, the summation is over all the possible Bloch wall configurations, and \( a(\{i, j\}) \) is the Boltzmann factor for the interaction between up spin \( i \) and down spin \( j \). For the symmetric nearest neighbor Ising model (in any dimensional space), \( a = e^{-2J} \).

Perhaps the smartest proof (based on the idea of Asano’s) of the above assertion is as follows.\textsuperscript{144}

The most important observation is that the polynomial (27.16) can be constructed recursively as follows.

1. Let
   \[
p(i, j) = z_i z_j + a(z_i + z_j) + 1. \tag{27.17}
   \]
   Notice that this is the partition function for the Ising spin pair, if we set \( z_i = z_j = z \) (\( z_i \) implies that spin \( i \) is down).
2. We multiply all the \( p(i, j) \) corresponding to all the nearest neighbor pairs.\textsuperscript{145}
   1. This product can be constructed successively one interaction pair by one.
   2. When the same \( z_j \)'s appear in this process, we perform the Asano contraction. Let \( P_X \) be the polynomial already constructed up to this point, and we wish to multiply \( p(i, j) \). Assume that \( j \) is in \( X \) (that is, \( z_j \) is shared by \( p \) and \( P_X \)). Suppose the formal product may be written as
   \[
P_X p(i, j) = A z_j^2 + B z_j + C. \tag{27.18}
   \]
   Then, the result of the Asano contraction is
   \[
   \rightarrow A z_j + C. \tag{27.19}
   \]


\textsuperscript{145}This step may be easily extended to non-nearest neighbor cases.
This is illustrated in Fig 27.1. In (27.18) $z_j^2$ implies that spin $j$ in $X$ and that in the pair $(i, j)$ are both down. The term $C$ means both are up, so the terms with $A$ and $C$ are consistent and realizable. However, the term with $B$ describes inconsistent situations that only one of $j$’s in $X$ or in the pair is down. Therefore, such a term must be purged. That is the meaning of the Asano contraction.

A simple example is (Fig 27.2)

$$p(1, 2)p(1, 3) = (z_1z_2 + a(z_1 + z_2) + 1)(z_1z_3 + a(z_1 + z_3) + 1) \rightarrow z_1(z_2z_3 + az_3 + az_2 + a^2) + (az_2 + 1)(az_3 + 1).$$

(27.20)

The result is indeed the partition function for the $L$ shaped 3 lattice point system.

Now, the general theorem can be restated as:

The polynomial constructed recursively according to the method explained above has no zero in the unit disk.

The demonstration is recursive. Let all the polynomials below be constructed as above with all the $z_i$ set equal to $z$.

1. $p(i, j)$ has no zero in the disk. This product can be checked explicitly.
2. Let $X$ and $Y$ be disjoint lattice point sets, and $P_X$ and $P_Y$ are constructed recursively as mentioned above. If $P_X$ and $P_Y$ do not have any zero in the disk, then neither does the product $P_XP_Y$. This is obvious.
3. The polynomial constructed from $P_Xp(i, j)$ with $j \in X$ and $i \notin X$ by the
Asano contraction has no zero in the disk. This is the only nontrivial part. Before equating all \( z_i \) with \( z \), we may write

\[
P_Xp(i, j) = Az_j^2 + Bz_j + C. \tag{27.21}
\]

If \( |z_i| < 1 \) for \( i \in X \), then this cannot vanish for \( |z_j| < 1 \). Thus, in this case \( C/A \) (the product of two roots of (27.21)) must have the modulus not less than 1: \( |C/A| \geq 1 \). Therefore, \( Az + C \) cannot have a root in the unit disk, either.

Now, the polynomial is symmetric under \( z \to 1/z \), so it cannot have any root outside the unit disk, either. Therefore, all the roots must be on the unit disk.

In the thermodynamic limit, we expect that the distribution of the zeros on the unit disk converges to some limit distribution. This can be shown with a theorem due to Wintner.\(^{146}\)

\(^{146}\)Lee and Yang thank Kac for showing them the proof.