Detecting strange attractors in turbulence
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$y : M \to \mathbb{R}$ is a smooth function (observable). What can you know about φ_t on M from y?

Theorem 1 Let M be compact and m-dimensional. $\varphi \in \mathrm{Diff}(M)$ and $y : M \to \mathbb{R}$ smooth. Then, it is a generic property that the map $\Phi : M \to \mathbb{R}^{2m+1}$:

$$\Phi(x) = (y(x), y(\varphi(x)), \cdots, y(\varphi^{2m+1}(x))) \tag{0.0.1}$$

is an embedding at least C^2.
Technical conditions further required:
(i) If x is periodic, its period is less than $2m+1$, all eigenvalues of $d\varphi^k$ are different ad distinct from 1.
(i) No two fixed points of φ gives the same y.
(iii) For Φ to be an immersion near a fixed point x the covectors $dy, d(y\varphi), \cdots d(y\varphi^{2m})$ span $T^*_x(M)$.[1]
Generically, we assume $\Phi(x)$ restricted to a compact V of periodic points is an embedding whenever (φ, y) is in a small set U. We wish to show that for some $(\varphi, y) \in U$ Φ is embedding.

First we show this is immersion, and then show it is injective. At any x the components of $dy = dx_i \frac{\partial y}{\partial x_i}$ may be perturbed independently, so immersion may be realized. Next, we make Φ injective by perturbation.
https://www.youtube.com/watch?v=6i57udsPKms&frags=pl%2Cwn good

Taken's paper

0.0.1 Differential topological rudiments

M and N are manifolds and $\dim M = m < n = \dim N$. If $F : M \to N$ is C^1, then $N \setminus f(M)$ is dense in N.
M and N are manifolds and $\dim M = m > n = \dim N$. If $F : M \to N$ is C^1, and submersive[2] at every point, then $F^{-1}(p)$ is a submanifold in M with dimension $m - n$.

0.0.2 Takens' embedding theorem

Let M be a compact manifold of dimension m. Let $\phi \in \mathrm{Diff}^2(M)$ (dynamics) and $y : M \to \mathbb{R}$ be C^2 (observable). Then, $\Phi_{(\phi,y)} : M \to \mathbb{R}^{2m+1}$ defined as

$$\Phi_{(\phi,y)} = (y(x), y(\phi(x)), \cdots, y(\phi^m(x))). \tag{0.0.2}$$

is, generically,[3] an embedding.

A version for a given diffeo ϕ.
Let $\phi \in$Diff(M) with
(i) only finitely many periodic points of period less than or equal to $2m$

[1] The condition on dy means that $dy = dx_i \frac{\partial y}{\partial x_i}$
[2] $DF : T_pM \to T_{F(p)}N$ is surjective.
[3] open dense

(ii) At periodic points of period k thje eigenvalues of ϕ^k are all distinct.
Then, $\Phi_{(\phi,y)}$ is embedding.

0.0.3 Obvious continuity properties

$\mathcal{F} : y \to \Phi_{(\phi,y)}$ is continuous.

0.0.4 Obvious openness

Let $K \subset M$ be compact. The set of y such that $\Phi_{(\phi,y)}$ is immersive[4] (or injectively so) on K is open in $C^2(M, \mathbb{R})$).

0.0.5 Deneseness proof

Given $y \in C^2(M, \mathbb{R})$, in its any nbh is y' such that $\Phi_{(\phi,y')}$ is an embedding of M.

Note first that **0.0.4** implies that if y is immersive, then y' in its sufficiently small nbh are all immersive. That is, (injective) immersiveness is not destroyed by perturbation.
(i) Periodic orbits of small periods make $\Phi_{(\phi,y)}$ degenerate. However, if there is only finitely many undesirable periodic orbits we can kill them with arbitrarily small perturbation of y. This is not enough; we must maintain y to be immersive at all the periodic points. $D(y(\phi^s)) = DyD(\phi^s)$ must have rank = dim M. This is possible only if $D(\phi^s)$ has this property.
(ii) Make Φ immersive. M is separated into a finite number of parts, so that the map is basically $\mathbb{R}^{2m}$ into $\mathbb{R}^{2m+1}$. (chart to chart). Perturb y on each patch.
(iii) Make Φ embedding on orbital segments. Let us call $o = \{x, \phi x, \cdots, \phi^{2m} x\}$ the orbital segment of x. If there is another orbital segment o' of x' that overlaps with o, it could make period $4m$ orbits. Thus, $o_2 = \{x, \phi x, \cdots, \phi^{4m} x\}$ and make all the points of the orbital segments are separated by a nbh X_x of x such that $\phi^j(X_x)$ $(j = 0, 1, \cdots 4m)$ are disjoint. Next, Since M is compact, choose a finite cover that can separate all the orbital segments.
(iv) injective immersion on M.

[4] $DF : T_p M \to T_{F(p)} N$ is injective (F need not be). That is, rank$DF = \dim M$.