Story Line

Here is a flow of story of Phys510 Fall 2018. Perhaps 6 semester materials are pushed into one semester, so I give here an overall outline of the flow, and summarize some important concepts and facts (theoretical) physicists should know. Key concepts are in boldfaces; you can look up the units relevant to the concepts in the main lecture notes by clicking the unit numbers (if you use the Story Line appended to the main body).
v

The course consists of four parts I - IV:
I. Introductory review of differential equations and maps (Lect 1-16).
II. Typical 'chaotic systems' and famous dynamical system (Lect 17-22).
III. Conceptual tools to understand dynamical systems (Lect 23-36).
IV. Outline of the modern theory: Peixoto to Palis (Lect 37-44).

Underlined statements are (my) conjectures perhaps theoretical physicists could look into.

Part Ia: non-conserved systems.

[**1**] Study of deterministic time evolution is the theory of dynamical systems (see **1.1**). Usually, we study **flows** defined by (sufficiently differentiable) **vector fields** on manifolds **2.5**, and **endomorphisms** (into-maps) defined on manifolds **2.2**. In many important cases, a **Poincaré map** **6.3** and its **suspension** **6.8** relate the continuous-time and its discrete-time descriptions.

[**2**] Theory of dynamical systems uses standard differential topological and geometrical terminologies freely. Therefore, to read the original math papers often demands some familiarity to differential topology. If you have no ambition to write math papers, intuitive understanding of the related concepts and theorems is sufficient (see **2.3**-**2.9**). However, the expression of tangent vectors in terms of ∂_i (**2.5**, **2.7**) is highly useful.

[**3**] As fundamental scientists we wish to have a general 'universal' or 'unified' understanding of many things, so we must properly characterize what we mean by 'general', '**generic**' (= residual **2.28**), etc. This is, however, not very simple, because simple characterizations are plagued with exceptions, and 'air-tight' characterizations tend to be cumbersome. Therefore, we confine ourselves to the study of systems stable against various perturbations (structurally stable systems; **2.13**). If the dimension

of the base manifold is not too large (2 or 3), then structural stable systems are quite numerous (generic or even sometimes open generic).

[**4**] Crudely put, **structurally stable systems** are characterized by **hyperbolicity** (**5.7**, **5.8**) and **transversality** (= relations are not 'tangential' or critical).

[**5**] Just as phase transitions, when qualitatively different stable features switch, structural stability is lost and systems become unstable against perturbations. This phenomenon is called **bifurcation** Lect 7, 8) **7.1**. Therefore, the theory of dynamical systems studies generic structural stable features and bifurcations exhibited by a collection of systems.

[**6**] There are two major ways to study dynamical systems, (i) **topological** and (ii) **measure-theoretical**. (i) is geometrical; we are interested in how trajectories go around (topologically). (ii) is statistical; we are interested in the average behaviors.

[**7**] Thus, our story line for 'Part I' goes as follows: In physics Hamiltonian dynamical systems are quite important. However, from the general dynamics point of view they are very special with the canonical structure. Therefore, first we discuss general ODE/Diffeo and their elementary bifurcations (Lect 3-9). Then, we go to Hamiltonian systems including elementary celestial mechanics (Lect 10-16).

[**8**] ODE (its origin: **3.26**) with continuous vector fields define flows **3.7**. Peano noted, however, that the uniqueness of the solution to initial value problems is not guaranteed as counterexamples show (**3.13**). Eyink points out such vector fields can be realized in fully developed turbulence as the flow velocity field (**3.14**). The spirit of the proof of **Peano's theorem** (**3.12**) with the aid of Arzela's compactness theorem (**3.11**) should be understood: we construct a sequence of approximate solutions, the totality of which makes a compact set, so we can find a limit which is a solution.

[**9**] The uniqueness of the flow defined by a vector field is guaranteed if the field is **Lipshitz** (Cauchy-Lipshitz theorem **3.18**). The reason may be intuitively understandable from the local **rectifiability** (**3.19**) of the field and its extension (**3.20**). The continuous dependence of the solution on its initial condition can be shown (**3.23**) almost constructively; **Gronwall's inequality** (**3.22**) is a standard tool.

[**10**] The uniqueness theorem breaks down at singularities where vector fields vanish. If the derivative at the singularity is non-singular, we say the singularity is simple

(**simply singularity 4.2**); simple singularities are isolated (**4.3**). We **linearize** the ODE around its isolated singular point; the solution to the linearized equation may be written in terms of a matrix A as (**4.5**)

$$\dot{x} = Ax.$$

[**11**] Its solution may be computed in terms of the evolution operator e^{tA} **4.6**. Constructing the (real or complex) Jordan form of A (Appendix 1 to Lect 4 for the general theory) is the standard way to compute this operator explicitly as a matrix; see a detailed example **4.17**.

[**12**] If the base manifold is a 2-manifold (2D manifold), then we can illustrate all the types of simple singularities: sink, source, saddle, focus, center **4.10**.

[**13**] On a given manifold not every vector field can live happily. There must be a topological consistency (**Poincaré-Hopf theorem 4.21**): the Euler characteristics of the manifold must be consistent with the sum of indices of the field (the **degree** of the field **4.20**).

[**14**] If A in [10] has no eigenvalue with vanishing real part, we say the isolated singularity x is **hyperbolic** (called a hyperbolic fixed point **5.1**). The stable (resp., unstable) eigenspace E^s (resp., E^u) is the subspace of T_xM (**2.5**) on which A has eigenvalues whose real parts are negative (resp., positive) (**5.7**). There is an invariant submanifold called **stable manifold** W_x^s (resp., **unstable manifold** W_x^u) of M on which the flow is contracting (resp., expanding) [**Stable manifold theorem 5.11**].

The renormalization-group flow near the critical point is a typical hyperbolic flow (**5.12**).

[**15**] Near a hyperbolic fixed point, the original dynamics and the linearized dynamics are homeomorphic (**Hartman's theorem 5.13**). Its proof is nontrivial, but it uses very standard functional-analytic tools; the basic idea of the proof is to construct the homeomorphism. The theorem justifies the **linear stability analysis** of a vector field around a hyperbolic fixed point. [See **5.20** and **5.21** for definitions of stability.]

[**16**] When bifurcation occurs, A is non-hyperbolic. Then, there is a neutral subspace on which eigenvalues of A have no real part. There is a manifold tangent to this subspace called the **center manifold** (not necessarily unique). The **center manifold theorem 5.19** allows us to make a reduced dynamics that is often with a smaller number of variables than the original (thus practically useful).

[17] For ODEs solution curves can be one of the following three: point, ring or line (**6.1**). The ring corresponds to a **periodic orbit**, for which we can construct a **Poincaré map** **6.3**.

[18] To study the stability of a periodic orbit, we linearize its Poincaré map (**6.4**). The resultant matrix is periodic, so we may use **Floquet's theorem** to isolate the non-periodic components (Floquet multiplier or Lyapunov constant **6.5**) to study stability.

[19] Isolated periodic orbits are called **limit cycles** **6.6**. For S^2 (or the domain embeddable in S^2) **Poincaré-Bendixson theorem** **6.9** tells us the existence of periodic orbits. In practice, the use of null-clines (**6.17**) may also be useful.

[20] When singularities are not hyperbolic, **bifurcations** **7.1** occur. To study systematically what can actually happen at or around bifurcation points, we make a 'standard form' (**normal form** **7.6**) of the field at the bifurcation point, and then consider its most general deformation (unfolding). This is the **versal unfolding** (**7.3**) approach. Its first step is to make the lowest order nontrivial normal form using the cokernel technique (**7.7**, normal form theorem **7.9**). Look at **Hopf bifurcation** as an example (**7.14**).

[21] Normal form analogue can be constructed for maps as well. Accumulation of 2^n-periodic orbit for a continuous endomorphism (See Sarkovskii's Theorem 3 in **22.26**) is an accumulation of pitchfork bifurcations **Feigenbaum critical phenomena** **8.7**). Feigenbaum constructed an RG theory (**8.7**-) to study this critical point. As you will see Sinai's thermodynamic formalism tells us the correspondence of this point and the critical phenomenon for 1D Ising model (with long-range interactions) (Lect 36 **36.1**-).

[22] The modifications appearing in versal unfoldings are perturbations that give qualitative changes to the system dynamics. That is, the perturbation series for such perturbations cannot converge. Such perturbations are called **singular perturbations** **9.1**. However, there is a way to obtain the long-time behavior of the perturbed system systematically. They are collectively called singular perturbation theory, many of which may be unified as a renormalization group theory (**9.10**).

[23] The most important observation in the RG approach to singular perturbations is that the RG equation is the slow time equation that describes the long-time effect of singular perturbation. If the method is applied to PDE's, very often the lowest nontrivial order RG equations are the 'named' equations, e.g., Boltzmann equation, Burgers equation, Swift-Hohenberg equation, etc. The reliability of RG equations is demonstrated by Chiba **9.18**. My conjecture is: if the original perturbed system is structurally stable, then the corresponding RG result is homeomorphic to it.[465]

Part Ib: conserved systems = Hamiltonian systems

[**24**] The **Newton-Laplace determinacy 10.1** and its compatibility with a **variational principle** (Veinberg's theorem **10.3**) imply that the equation of motion is a conserved second-order time-reversal symmetric equation. The **action principle** is locally a minimum principle **10.5**.

[**25**] A Legendre transformation of Lagrangian gives the Hamiltonian, and the action principle is rewritten as **Hamilton's principle 10.9**. In terms of **Poisson brackets 10.10** the Newton's equation of motion can be written symmetrically as the **canonical equation of motion 10.11**. Jacobi's identity may be demonstrated easily (**10.12**), if we introduce the infinitesimal canonical transformations (**13.5**).

[**26**] If a system with n-degrees of freedom[466] has a set of n independent invariants, then we say the system is **completely integrable 11.4**. Then, the phase space is foliated into nested invariant T^n (**Liouville-Arnold's theorem 11.6**; its demonstration is not so trivial as seen in **11.8**). Each torus is specified by the values of action variables, and the motion on it is described in terms of the angle variables **11.7**.

[**27**] Most (all?) completely integrable systems may be expressed in terms of a **Lax pair** A and L as (**12.1**)

$$\dot{L} = [A, L].$$

The eigenvalues of L are the invariants. The Toda lattice **12.3** is an example, which is related to the Kortweg-de Vries equation (**12.6**; a (not terribly) quick and dirty derivation **12.8**), which is famous for exhibiting solitons. Initially, the equation drew attention for its closeness to the Fermi-Ulam-Pasta problem **12.5** (but actually, not so close).

[**28**] In mechanics we consider only canonical transformations with **generators 13.1**. If the transformation is infinitesimal, it is called an **infinitesimal canonical trans-**

[465] The discrete time counterpart of this statement is a theorem.

[466] i.e., with n functionally independent variable pairs $\{q_i, p_i\}$

formation **13.4**. Time evolution is an example, whose generator is the Hamiltonian (**13.6**). Noether's theorem may be understood with its aid (**13.7**).

[**29**] To show the invariance of Poisson brackets under canonical transformations, we introduce **Lagrange brackets** **13.9**. This machinery allows us to prove **Liouville's theorem** **13.15**. Also we can show that Poincaré maps preserve cross-sectional areas (**13.11**) for Hamiltonian systems.

[**30**] Note how special the Newtonian potential is (Bertrand's theorem **14.1**). It is almost impossible to show the stability (no collision, no escape) of $n(> 2)$-celestial body system theoretically (cf. **14.3**).

[**31**] Even the **restricted three body problem** **14.6** is too complicated to study analytically. Poincaré showed that there is no integrable of motion functionally independent of the Hamiltonian that is analytic in the perturbation parameter (**14.8**). Thus almost all lost interest in solving the restricted problem.

[**32**] However, there are two stable fixed point solutions (Lagrangian points; from the SJ co-rotating coordinate system). These points describe Trojan asteroid group (**14.14** and more with respect to the earth and the moon **14.15**).

[**33**] Although Poincaré realized how complicated the three-body problem is (see **16.3** and the figure), Kolmogorov realized that still many invariant tori (esp highly non-resonating orbits **KAM tori** examples in Lect 16) guaranteed by the Liouville-Arnold theorem survive. There are two obstacles to prove the assertion. One is the **small denominator problem**, which was overcome by Siegel (Siegel's stability theorem **15.12**) with the so-called Diophantine approximation (**15.11**). The other is to prove the actual convergence of the perturbation series. The basic idea for the latter was furnished by Kolmogorov by 'partial linearization' (see **15.13**; **15.25**).

[**34**] What happens if the tori are deformed? This may be glimpsed from Poincaré-Birkhoff's theorem **16.8**. We have elliptic and hyperbolic periodic orbits, and the possible heteroclinic orbits produce chaos as shown in **16.9**. In the chaotic region the system can wander off far away from the original torus (especially if the system is high-dimensional; called **Arnold diffusion** **16.13**)

[**35**] The FPU system does not thermalize due to the persistence of the KAM tori as noted in **16.12**. Motion of charged particles in electromagnetic fields is an important topic from the accelerator physics and plasma physics. A typical simple case is illustrated in **16.4** and may be understood in terms of the standard map **16.10**.

Part II: The Zoo

Here is a flow of story for Lectures 17-22. This is a showcase of representative examples, billiards, coupled relaxation oscillators, Lorenz system, Ruelle-Takens picture/strange attractors, interval endomorphisms + related concepts and theorems.

[**1**] Perhaps the simplest Hamiltonian system is a ballistically moving particle perfectly elastically colliding with boundaries/obstacles. Usually we discuss such systems defined on a 2-flat space. They are generally called **billiards**. Their overall dynamics may be understood from the mean free time and what happens at collisions (**Ambrose-Kakutani representation** **17.2**; **17.15**, mean-free time **17.16**; **Abramov formula** **17.17**).

[**2**] Noteworthy facts about billiards include:
(1) Even on polygons (triangles) a lot of things are not yet understood. See **17.4**.
(2) If the table is convex and if the boundary is sufficiently smooth, there is a **caustic**, so the system cannot be fully chaotic (even if chaotic) (Lazutkin **17.6**).
(3) **Sinai billiards** (dispersive billiards **17.7**) are 'maximally chaotic.'[467] Often they have, at least conceptually, related to geodesics on negative curvature surfaces **17.12**. These billiards are chaotic (intuitively), because information is lost upon collisions **17.18**. Some details about computing information loss rate (the Kolmogorov-Sinai entropy) is explained towards the end of Lecture 17.
(4) **Bunimovich billiards** (converging billiards) are also fully chaotic **17.13** [but the shapes are rather restricted (why? cf Lazutkin above)].

[**3**] Mutually hindering coupled relaxation oscillators are chaotic **18.1**. Methods to analyze such systems (converting to geometrical models and maps (e.g., **18.7**, **19.4**) are standard techniques). We can easily understand why the system is not predictable **18.5**.

Coupled relaxation oscillators can be related to the **Lorenz system** **19.3**; the motivation with its relation to the Rayleigh-Benard problem (Saltzman's equation **19.2**) is explained in **19.1**-; the reduction method is an example of the Galerkin method **20.8**.

[467] They are Bernoulli systems.

A very similar model is the Rikitake model of the earth dynamo **19.7**.

[**4**] Mathematical properties of the Lorenz model are highly nontrivial to study; even to establish the existence of the nontrivial attractor is not easy **20.3**. The existence of a physically observable invariant measure (introduction **29.4**) is hard to prove, although the binary Ising spin coding of dynamics on the Lorenz attractor by Shimada **20.7** was very suggestive.

Therefore, mathematically more transparent **geometrical models 20.2/templates 20.6** were studied. The latter are used to establish the existence of knotted orbits **20.1**.

The Lorenz system is not the usual chaotic system. For example, it is very likely to lack the tracing property **20.5**.

[**5**] The idea of the **strange attractor** (Definition, for example, **21.5**) was introduced by **Ruelle and Takens 21.2** to demonstrate that scenarios different from Landau's leading to turbulent flows exist. They constructed an example of a flow in T^4 and later in T^3 **21.3** (but actually observing them even numerically is almost impossible **21.4**).

[**6**] An endomorphism of an interval was used by Lorenz to show that his result is not due to simple numerical errors **19.4**. Also May pointed out such simple systems exhibit chaotic behaviors **22.1**. Li and Yorke published a paper, "Period three implies chaos' **22.2**. The simplest example of the interval map is illustrated in detail in **22.4** (you understand the essence of chaos if you understand this unit). Since I did not like Li-Yorke chaos, I introduced a more natural definition **22.10** equivalent to now popular definitions and showed necessary and sufficient conditions (e.g., "Period $\neq 2^n$ implies chaos") for a C^0-endomorphism to exhibit chaos **22.14**, **22.15**.

For periodic orbits of a C^0-endomorphism of an interval **Sarkovski's theorem 22.26** tells us the universal ordering of the appearance of periods.

[**7**] Other famous systems show up with more general discussion: baker's transformation **27.1**, horseshoe **28.1**, Bernoulli shift **34.3**, etc.

Part III: Conceptual tools

The portion is 23-36, which is more or less conceptual: symbolic dynamics, algorithmic randomness, Brudno's theorem, baker's transformation and horseshoes, ergodicity, entropy, Lyapunov indices, thermodynamic formalism, etc.

[**1**] What is the most natural characterization of chaotic dynamical systems (see **24.8**)? My intuition is: if (observable) orbits have natural relation to (say, after an appropriate coding) random number sequences, the system is chaotic.

To make this statement meaningful, we need precise mathematization (conceptual analysis) of 'randomness.' To this end algorithmic random numbers are introduced **23.23**; this requires clarification of algorithm and **computation**. Thus we have to go all the way back to **Church** **23.6**-**23.14** and **Turing** **23.16**-**23.19**.

[**2**] We use the most powerful machine **UTM** **23.20** and compress the number sequence. If you cannot compress it significantly, the sequence is random **23.23**. Roughly speaking, when we discretize a system along the time axis (say, with the aid of the Poincaré map **6.3**), the needed length of the shortest program to reproduce the code sequence divided by its duration time (the length of the sequence) is the **complexity of the trajectory**.

[**3**] If we can make faithful mapping (homomorphism) of a dynamical system to a shift (introduced in **22.6**; more formally **26.1**), we use the latter as a code sequence to analyze the trajectory. If we cannot information-compress it, then the sequence is random and the trajectory is chaotic.

One problem is that there is no way to judge whether a given sequence is random or not generally **23.24**, but collectively we can say, e.g., a set consists of mostly random numbers (for example, we can say that binary expansion of $\omega \in [0,1]$ almost surely gives a random number).

[4] **Brudno's theorem** **24.4** tells us that the Kolmogorov-Sinai entropy (informally introduced in **17.18**, **17.19**) of a (measure-theoretical) dynamical system is identical to the (average) complexity of the trajectories.

This is probably the best characterization of chaos, or chaotic dynamical system at least for measure-theoretical systems (informally **1.2**; Lect 29, esp **29.1**). (If no measure is introduced, we can say a dynamical system is chaotic, if its topological entropy **32.19** is positive.)

[**5**] Whether a dynamical system itself is computable (e.g., can we compute the trajectory position at time 10?) is usually not discussed, but if a theory is a part of physics, its outcome must be compared with observations. If we demand some quantitative agreements, we must be able to compute the numerical outcomes of a theory. Thus the question whether the answer is numerically computable becomes a crucial question (**computable analysis** Lect 25). The prediction must be given in terms of computable reals (**25.13**, effective limits **25.12** of computable rational sequences **25.11**).

A noteworthy point is that even if a function is twice differentiable, its second derivative may not be numerically evaluated **25.21**. What is its implication in physics (say, Newton's equation of motion)?

[**6**] Although a time-discrete dynamical system always has a time continuous counterpart (constructed by suspension) whose Poincaré map can give the original system, a discretized time-continuous system may not be able to recover the original time-continuous system. However, for almost all natural systems we may go back and forth freely between the two descriptions (especially for statistical behaviors). Thus, study of symbol sequences or symbolic dynamics is quite important (as we have already seen in [III4]). Formally they are **shift dynamical systems 26.2**. Its subclass called **Markov subshifts 26.6** is quite important. Shift dynamical systems may be interpreted as 1D lattice equilibrium statistical mechanical models (entropy per spin = the KS entropy, for example) [Thermodynamic formalism]. Consequently the theory of Gibbs measures **36.5** becomes crucial. Since it is 1D the transfer matrix **36.6** is important, and the Perron-Frobenius theorem is a key (**26.11** or **35.10**).

[**7**] A typical use of symbolic dynamics is illustrated with the aid of **baker's transformation 27.1** and **Smale's horseshoe 28.1**. Horseshoes appear everywhere we see chaotic behavior; as we can see from Poincaré's celestial mechanical studies **16.3** homo- and heteroclinic crossings are everywhere (e.g., see **16.9**).

[**8**] For a given dynamical system usually there are infinitely (very often uncountably many) distinct **invariant measures 29.5**, **29.6** (also a summary: **36.1**; For "What is measure?" see **29.12**-). For each invariant measure, that may be interpreted as a particular stationary state of the underlying dynamical system, we can make a measure-theoretical dynamical system **29.8**.

In physics observability is of superb importance. If an invariant measure is absolutely continuous **29.9**, it is very likely to be observable (numerically, or in actual experiments).

[**9**] **Ergodicity 29.10** and **mixing property 29.11** are properties of measure-theoretical dynamical systems, so invariant measures must be explicitly specified; topological transitivity and mixing **26.4** are topological counterparts of these concepts, but when we are interested in expectation values as in statistical mechanics relevant concepts are always measure-theoretical.

[**10**] Thus, when one says a system is ergodic in classical statistical mechanics, one means the Liouville measure is ergodic; this is almost never proved for any interesting systems. Although it is an irrelevant question for the foundation of statistical

thermodynamics, even it were relevant, we must note that an invariant measure is selected by the initial condition, so the invariant measure is subordinate to the sampling measure of the initial conditions. Everybody knows that the choice of the initial condition has nothing to do with the system dynamics. This clearly tells us the meaninglessness of the ergodicity question in statistical mechanics.

[**11**] The most important theorems related to the system ergodicity is **Birkhoff's ergodic theorem** 30.4 and **Poincaré's recurrence theorem** 30.2. Zermelo used the latter to unravel Boltzmann's logical weak point in his second law argument **30.3** (see also its tragicomical history **30.11**). Note that Birkhoff's theorem has no direct relation to the ergodicity of the system (read the original theorem statements).

[**12**] We know already (see [III4]) the superb importance of information in understanding dynamical systems. An intuitive approach to the **Kolmogorov-Sinai entropy** **31.2**, Rokhlin's formula, etc., as well as a formal definition through partitions may be found in Lect 32 (e.g., **32.6**). We use a special partition called the generator **32.10**.

Around here why information is quantified by Shannon's formula is explained through Sanov's theorem **31.8**.

[**13**] Krieger's theorem **32.12** about coding of a trajectory anticipates Brudno's theorem. Practically, the most important theorem is the **Shannon-McMillan-Breiman theorem** **32.13** which states the relation between the size (measure) of the elements of the partition and the KS entropy. The theorem applied to Bernoulli systems is the **asymptotic equipartition theorem** **32.14** that justifies the principle of equal probability.

[**14**] Chaotic systems exhibit orbit instability: nearby orbits separate from each other exponentially. This extent (or the parting rate) is measured by the **Lyapunov characteristic number** (LCN) or indices **33.1** (related to the Lyapunov exponent for periodic orbits **6.5**). **Oseledec's multiplicative ergodic theorem** **33.2** guarantees their existence and initial-condition independence (if the system is ergodic); this theorem may be most conveniently proved with the aid of **Kingman's subadditive ergodic theorem** **33.8**.

[**15**] LCN is closely related to the KS entropy as expected from the Shannon-McMillan-Breiman theorem; if the system is sufficiently smooth, then the sum of positive LCN is equal to the KS entropy (**Pesin's theorem** **33.6**).

[**16**] Introduction of information into dynamical systems by Kolmogorov as an **iso-**

morphism invariant 34.1 led to an outstanding question: is it **complete**? This culminated in **Sinai's and Ornstein's theorems** 34.5: For Bernoulli systems 34.3 the KS entropy is a complete invariant.

The original proof is very constructive (and long), but recent 'soft-proofs' are much shorter.

[**17**] We can study empirical time averages of observables as a function of the time span 35.1, 35.2 (the **large deviation approach**). Extending Sanov's theorem to the current situation 35.6, we can derive Rokhlin's theorem 31.2 and Pesin's theorem 33.6.

[**18**] Since the code sequences and spin configurations on 1-lattices are one-to-one correspondent, Sinai introduced the **thermodynamic formalism**, in which entropy per spin = the KS entropy. This is closely related to the Fredholm theory of the Perron-Frobenius equation 36.11. Even we could introduce temperature 36.16 that seems to be related to the Hausdorff dimension 2.25 of the support of the invariant measure.

In terms of the Fredholm determinant an outstanding conjecture may be 36.18: the multiplicity of eigenvalue 1 is the number of distinct physical invariant measures 36.14 in the Kolmogorov sense (= stability against adding noises).

This approach tells us 36.13 that log of the expansion rate of the unstable manifold corresponds to the Hamiltonian.

[**19**] How can we observe strange attractors? This is answered by **Takens' embedding theorem** 37.1. For example, plotting $(2n+1)$-vectors consisting of consecutively obtained $2n+1$ observed values of a scalar observable in $2n+1$-space, generically we can reconstruct the n-strange attractor of the system. The theorem is experimentally useful. Lect 37 is grossly incomplete: I believe a bit more restrictive but intuitively provable (i.e., that theoretical physicists can 'prove') statement is possible, but is not yet written up.

Part IV: from Peixoto to Palis

The last part is an outline of the modern theory of dynamical systems from Peixoto to Palis: Morse-Smale systems, Axiom A and Anosov systems and technical tools such as shadowing and Markov partition and SRB measures. Palis' conjecture about what we can find in the world conclude the lectures.

[**1**] For C^1 vector fields on 2-manifolds (differentiable), Peixoto asked a necessary

and sufficient condition for the structural stability **38.2** of the flows. Peixoto proved (with his wife) **Peixoto's theorem 38.3**: A vector field $X \in \mathcal{X}^1(M)$ is structurally stable if and only if: (i) there are only finitely many singularities (all hyperbolic), (ii) the limit set consists of fixed points or hyperbolic limit cycles (iii) without any saddle connections.

[**2**] The structural stability of X satisfying these condition (that is, the sufficiency part **38.7**) is proved explicitly classifying the possible 'patches of X and studying all of them one by one.

The sufficiency part **38.14** is proved through showing that any X may be converted to a field satisfying the itemized conditions above with arbitrarily small perturbations. If the original field X is structurally stable, it must have had the same features as after the perturbation.

[**3**] Peixoto's theorem was complete and clean, so it ignited interests of many good mathematicians including Smale. Smale wondered what happened on high-dimensional manifolds **38.4**. He defined the **Morse-Smale system 38.5**: (MS) Nonwandering sets consist of periodic points; (MS2) they are all hyperbolic; (MS3) the stable and unstable manifolds are always transversal. Peixoto's theorem may be stated: on 2-mfd flows are structurally stable iff MS.

[**4**] Since horseshoes can live on 2-mfd, it can be in a Poincaré map of a flow on 3-mfd, and since horseshoes are structurally stable (see Fig. 28.4), already on 3-mfd Peixoto's theorem does not hold.

[**5**] From Peixoto's theorem (and its proof) we see that hyperbolicity and transversality are crucial. Also horseshoes are structurally stable and appear 'everywhere,' Smale introduce another class of dynamical system **Axiom A 39.1**: a diffeo f satisfies Axiom A iff its nonwandering set $\Omega = \overline{\text{Periodic points of } f}$ and hyperbolic.

[**6**] The nonwandering set of an Axiom A diffeomorphism consists of invariant pieces (spectral decomposition theorem **39.8**). Intuitively (from the figures in the units) it is clear that we can introduce local 'canonical' coordinate systems consistent with local stable and unstable manifolds **39.6**. Using these coordinates, we can show the Axiom A systems have the **tracing property 39.13**. Using this and the local canonical coordinates, we can construct a **Markov partition 39.17**. In terms of this partition we can map M to a set of Markov subsequence, and map the original dynamical system isomorphically to a Markov subshift **39.22**.

[**7**] [This portion has not been explained.] Intuitively speaking the coding due to

the constructed Markov partition is 'local' in the sense that the distance between x and y in the real space is monotonically reflected on the closeness of the corresponding code sequences. In the original space the 'Hamiltonian' $-\log L_+$ is a function of the position (i.e., totally localized in space without any interaction across space) but dynamics may be strongly correlated. After coding, a function of space spreads over symbols, but due to the local nature of the map explained above, the new Hamiltonian is still short-ranged (decaying exponentially). Thus, we can use the usual 1D thermodynamics to make the Gibbs state, which mapped back to the original space is an absolutely continuous invariant measure, which is the **Sinai-Ruelle-Bowen (SRB) measure** (mentioned in **2.29**), (thanks to Birkhoff's theorem **30.4**) because it is ergodic.

This is the standard approach, but as a theoretical physicist, I wish to go directly from the free energy expression to the canonical distribution **36.13**.

[**8**] If $\Omega = M$ Axiom A systems are called **Anosov systems** **40.1**. The cleanest example is the linear toral diffeomorphisms (group automorphisms) **40.4** including the famous Thom's map (called erroneously Arnold's cat map) **40.5**. In these cases Markov partitions are rather easy to construct **40.6**.

[**9**] As an example of applications of dynamical systems to 'standard physics problems' self-similar spectrum of almost periodic 1d-lattice discrete Schrödinger problem is discussed. Although the example uses a rather acrobatic relation between the original and the dynamical system descriptions, the Cantor structure is exhibited without any room for doubt by the presence of Horseshoes or related structure **41.6**.

To try to understand difference equations as a diffeo problem is often useful.

[**10**] Initially, Smale thought Morse-Smale systems are sufficiently general dynamical systems (generic, hopefully open dense, but at least dense) in the totality of dynamical systems. This was only true on a 2-mfd as Peixoto's theorem indicates; His own horseshoe, that can live in a flow of T^3 and that is structurally stable, destroyed the idea that Morse-Smale is dense or open in any dimension higher than 2. Thus. Smale proposed Axiom A, and expected such systems are at least Ω-stable (i.e., the nonwandering sets are stable, even if the whole system is not).

[**11**] However, again very soon he realized that if there is a cycle **43.7** connecting the basic sets (see **39.8**) of the system, Axiom A systems cannot even Ω-stable: there is an Ω-explosion **43.8**, although such examples are no more dangerous with arbitrarily small perturbations. Thus, generic picture is intact.

[**12**] Then, came a surprise: Newhouse phenomenon: if there is a homoclinic or

heteroclinic tangency **42.2**, then there is an open set of dynamical systems that have infinitely many sinks **42.5**.

[**13**] This is shown by demonstrating two things (1) homoclinic tangency can be stable under any small perturbations and (2) if s system has a homoclinic tangency with arbitrary small perturbations it can be converted to a system with infinitely many sinks,

(1) is shown by crossing of stable and unstable foliations packed close to the tangent point (**42.6**-**42.10**). (2) is shown by studying how horseshoe emerges **42.13**.

[**14**] As we will see up to 2-mfd for maps (3-mfd for flows) the Newhouse phenomenon is the 'worst.' Then, a multidimensional extension of Smale's Ω-explosion example **43.8** was discovered to be made stable: the heterodimensional cycles **43.9**. This time, there is a set of systems with stable saddle connections. Thus, Axiom A + no cycle condition is not dense not open.

[**15**] A separate question is the characterization of structural stability, the Peixoto's original question.

Palis and Smale conjectured: Axiom A + strong transversality iff structural stability. For C^1 dynamical systems (both maps and flows) this is now a theorem (Robbin+Robinson, Mañe, Hayashi) **44.1**.

This means there is a chance for physicists to encounter structurally unstable systems, since they make an open set.

[**16**] As to the genericity question or the question about the 'common' systems we encounter in the world, Palis formulated the following conjecture (**Palis conjecture**) **44.3**:

1. Every C^r-diffeo of a compact mfd M can be C^r-approximated by one of the following:

(a) a hyperbolic system (Axiom A with strong transversality)

(b) a system with heterodimensional cycle **43.9**

(c) a system exhibiting a homoclinic tangency **42.2**.

2. If M is 2D (for maps), then (a) or (c) occurs (in other words, Palis conjectured that avoiding homoclinic bifurcation, Peixotos picture can be recovered in one dimension higher space).

2 has been proved for $r = 1$.

[**17**] The YouTube movie Chaos illustrates another Palis conjecture **44.2**: There is a C^r-dense set of dynamical systems with finitely many attractors whose union of basins of attraction has total probability. These attractors support physical mea-

sures.