9 Lecture 9: Singular perturbation and renormalization

9.1 Regular and singular perturbations[110]

Let us consider an ODE whose vector field is $X_0 \in \mathcal{X}^r(M)$. When we add a 'small' term εX_1 to X_0, where ε (> 0) is a small number, and $X_1 \in \mathcal{X}^r(M)$, the resultant systems is said to be a perturbed system

$$\dot{x} = X \equiv X_0 + \varepsilon X_1. \tag{9.1}$$

Perturbations may be classified broadly into two classes, regular and singular perturbations. Pragmatically speaking, if the result as a power series in ε is convergent (for some fixed time range independent of ε), the perturbation is regular; if not, singular. If the perturbation is regular, there is no qualitative change of the dynamical system due to perturbation. If singular, generally, the perturbed system exhibit new features, e.g., new appearance of limit cycles.

Singular perturbations may be classified into two major classes; X_0 is actually defined on a submanifold of M (in other words X_1 requires new vector components; this is probably the singular perturbation in the original sense) and X_0 and X have no such submanifold structure (e.g., the resonance problems).

Since the perturbation series formally obtained by a straightforward perturbation calculation for singular perturbation problems give (at best) asymptotic series just as interacting field theories, it may not be so surprising that renormalization-group

[110]The best reference book (practical book) of singular perturbation is C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, 1978). Numerous examples and numerical confirmation of the results make this book unrivaled. It is a book to be kept at one's side whenever singular perturbation problems are studied. However, the book is wonderfully devoid of any mathematical theory. To have an overview of various singular perturbation methods within a short time, E. J. Hinch, *Perturbation Methods* (Cambridge UP., Cambridge, 1991) is recommended. J. Kevorkian and J. D. Cole *Perturbation methods in applied mathematics* (Springer, 1981) was a standard reference.

ideas can be useful.[111,112]

9.2 Simple example of singular perturbation

Consider

$$\epsilon \frac{d^2 y}{dt^2} + \frac{dy}{dt} + y = 0, \tag{9.2}$$

where $\epsilon > 0$ is a small constant. If this is zero, the solution decays exponentially; Ae^{-t} is its general solution, where A is a numerical constant. If $\epsilon > 0$, for a sufficiently small time, the second order derivative term is important (notice that however small the mass may be, the inertial effect is crucial for a short time at the beginning of the motion). However, after a long time, the system behavior should be similar to the $\epsilon = 0$ case. Then, why don't we take this term into account through perturbation? This problem is easily solved by hand exactly, but let us pretend that we cannot do so, and perform a perturbative calculation.

9.3 Naive perturbation and its difficulty

Let us expand the solution to (9.2) formally as

$$y = y_0 + \epsilon y_1 + \cdots \tag{9.3}$$

and then introduce this into the equation (9.2). Equating the terms with the same power of ε, we obtain

$$\frac{dy_0}{dt} + y_0 \;\; = \;\; 0, \tag{9.4}$$

[111] Key ideas are in N. D. Goldenfeld, O. Martin and Y. Oono, "Intermediate asymptotics and renormalization group theory," J. Scientific Comp. **4**, 355 (1989); L.-Y. Chen, N. Goldenfeld, Y. Oono, and G. Paquette, "Selection, stability and renormalization," Physica A **204**, 111 (1993). The latter clearly recognized the relation between the reductive perturbation theory (fully exploited by Y. Kuramoto) and RG. Thus, the crux of many singular perturbation methods is to construct the equation of motion that governs the asymptotic solutions of perturbed systems. A paper with many examples is the 'CGO': Lin-Yuan Chen, Nigel Goldenfeld, and Y. Oono, Renormalization group and singular perturbations: Multiple scales, boundary layers, and reductive perturbation theory, Phys Rev E **54**, 376 (1996). Some misunderstandings (even by my collaborators) were corrected in Y. Oono and Y. Shiwa, Reductive renormalization of the phase-field crystal equation, Phys. Rev. E **86**, 061138 (2012).

[112] In these days, applied mathematicians have organized the results of the renormalization group method as a method of nonlinear variable transformation that requires no idea of renormalization. I have no interest in this direction.

$$\frac{dy_1}{dt} + y_1 = -\frac{d^2y_0}{dt^2}, \tag{9.5}$$

etc. Let us write the solution to the first equation as $y_0 = A_0e^{-t}$, where A_0 is an integration constant. Then, the general solution to the second equation reads

$$y_1 = A_1e^{-t} - A_0te^{-t}, \tag{9.6}$$

where A_1 is also an integration constant. Combining these two results, we obtain to order ϵ

$$y = A_0e^{-t} - \epsilon A_0te^{-t} + O(\epsilon^2). \tag{9.7}$$

Here, $A_0 + \epsilon A_1$ is redefined as A_0 (we ignore $\varepsilon^2 A_1$ in the second term; as can be seen from this, in the perturbative expansion we have only to find special solutions).

In this way, we can compute any higher order terms, but this is usually regarded as a bad solution, because ϵ appears with t which increases indefinitely. Thus, the perturbation effect that should be small becomes not small, and the perturbation method breaks down. In other words, the perturbation result may be used only for the time span much shorter than ϵ^{-1}. Mathematically speaking, the convergence is not uniform in time. The term with multiplicative t is traditionally called a *secular term*.

9.4 How we extract long time behaviors

The analogy with the standard RG problem (or critical phenomena) is as follows: We are interested in the long term behavior $t \to \infty$ that is insensitive to initial details (= the long-term qualitative behaviors). In this limit secular terms diverge. If we could remove such divergences, then naive perturbation series as obtained above could make sense.

For the present problem to watch the behavior just in front of us ('at present time') corresponds to macroscopic observations and the behaviors long ago correspond to microscopic scales.[113] That is, we are interested in the global behavior that does not change very much even if the initial condition is modified. In the ordinary renormalization problem, the microscopic-detail-sensitive responses (that diverge in the $L/\ell \to \infty$ limit, where L is our scale and ℓ the atomic scale) are separated and renormalized into materials constants.

[113] For ordinary deterministic systems it is hard to obtain the initial condition from the observation result at t (it is asymptotically impossible in the $t \to \infty$ limit). For chaotic systems, however, this is not only always possible, but the estimate of the initial condition becomes more accurate if we observe longer time asymptotic behaviors (however, we assume there is strictly no noise). That is, in a certain sense, chaos is the antipode of renormalizability.

Therefore, in the present problem what must be renormalized is the sensitively dependent behavior on the initial condition, and the place it should be pushed into must be the integration constants (the quantities connecting what we observe now and the initial condition); it is a natural observation, because the integration constants are determined by the initial condition.

Notice that what we can renormalize is the relation between what we can observe and what we cannot. Since there is no arbitrariness in the relationships among observable quantities, there is no room for renormalization constants for observable relationships. It is crucial to distinguish what we can observe and what we cannot.[114]

9.5 Renormalization along time axis

Separate the secular divergence as $(t - \tau) + \tau$, and then absorb τ by modifying unobservable A_0 (since we do not know the precise initial condition) as $A(\tau)$, which is understood as an adjustable parameter to be determined so that the solution agrees with the behavior we directly observe at present, i.e., around time t.[115] After this renormalization, the perturbation series (9.7) reads

$$y = A(\tau)e^{-t} - \epsilon(t - \tau)A(\tau)e^{-t} + O(\epsilon^2). \tag{9.8}$$

(As can be seen from this, τ is actually $\tau -$ initial time, i.e., the time lapse from the initial time). Such a series is called a renormalized perturbation series.

This equation makes sense only when $\epsilon(t - \tau)$ is small, but it is distinct from the original perturbation series (9.7) we started with (which is often called a 'bare' perturbation series), because we can choose τ to be large enough.

9.6 Renormalization-group equation

Since τ is a parameter not existing in the problem itself, $\partial y/\partial\tau = 0$. This is the renormalization group equation for the current problem:

$$\frac{\partial y}{\partial \tau} = \frac{dA}{d\tau}e^{-t} + \epsilon(t - \tau)\frac{dA}{d\tau} + \epsilon A e^{-t} + \cdots = 0. \tag{9.9}$$

The equation tells us that $dA/d\tau$ must be of order ϵ (the terms proportional to e^{-t} must cancel each other), so we may discard the second term of (9.9) as a higher order

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[114] Thus, the following ancient teaching becomes a crucial renormalization instruction: "When you know a thing, recognize that you know it, and when you do not, recognize that you do not." *Analects* Book 2, 17 [A. Waley, *The Analects of Confucius* (Vintage, 1989)].

[115] If you know the ordinary field-theoretic RG, you must have recognized that log(length scale) corresponds to time; $t - \tau \leftrightarrow \log(L/\ell)$.

term. Therefore, the renormalization group equation reads, to order ε,

$$\frac{dA}{d\tau} = -\epsilon A. \tag{9.10}$$

The renormalized perturbation series (9.8) is simplified if we set $\tau = t$ (those who question this procedure should see the note below):

$$y = A(t)e^{-t}. \tag{9.11}$$

(9.10) implies that $A(t)$ obeys the following amplitude equation:

$$\frac{dA(t)}{dt} = -\epsilon A(t). \tag{9.12}$$

The equation indicates that A changes significantly only in the long time scale of order $t \sim 1/\epsilon$; only when ϵt has a visible magnitude can A change significantly. Finally, solving (9.12), we get the following asymptotic behavior,

$$y = Be^{-(1+\epsilon)t} + O(\epsilon^2), \tag{9.13}$$

where B is an adjustable parameter. This is our conclusion about the asymptotic behavior.[116] Here, notice that the form of $A(t)$ as a function of t is universal in the sense that it does not directly depend on the initial condition ((9.12) is determined by the original differential equation itself).

As can be seen from the above example, the core of the renormalization group method for the problems with secular terms is to derive equations that govern the slow systematic motions such as (9.12). This may be interpreted as coarse-graining or reduction of the system behavior.

9.7 More systematic approach

To perform calculation more systematically, we introduce the renormalization constant Z as $A = ZA_0$ or more conveniently as (notice that the following Z is the reciprocal of the Z in $A = ZA_0$)

$$A = ZA_R, \tag{9.14}$$

where A is the 'bare' microscopic quantity and A_R the renormalized counterpart (in the current context it is what we observe long time later). We are performing a perturbation calculation, so we expand $Z = 1 + \epsilon Z_1 + \cdots$, and the coefficient are determined order by order to remove divergences. In the lowest order calculation as we have done, there is no danger of making any mistake, so a simple calculation as explained above is admissible, but, as we will see in Note 3.7.1, a formal expansion helps systematic studies.

[116] In terms of the two roots $\lambda_\pm = (-1 \pm \sqrt{1-4\epsilon})/2\epsilon$ of the characteristic equation $\epsilon s^2 + s + 1 = 0$ the analytic solution for (9.2) is $y(t) = Ae^{\lambda_+ t} + Be^{\lambda_- t}$, where for small ϵ $\lambda_+ = -1 - \epsilon + O(\epsilon^2)$, $\lambda_- = -1/\epsilon + 1 + \epsilon + O(\epsilon^2)$, so (9.13) is a uniformly correct order ε solution up to time $\epsilon t \sim 1$.

9.8 Why we can set $t = \tau$

In the above calculation putting $\tau = t$ makes everything simple, but there are people who feel that it is a bit too convenient and *ad hoc* a procedure, so let us avoid this procedure. The result of renormalized perturbation series has the following structure:

$$y(t) = f(t; \epsilon\tau) + \epsilon(t - \tau)g(t) + O(\epsilon^2). \tag{9.15}$$

Since f is differentiable with respect to the second variable, with the aid of Taylor's formula we may rewrite it as

$$y(t) = f(t; \epsilon t) + \epsilon(\tau - t)\partial_2 f(t, \epsilon t) + \epsilon(t - \tau)g(t) + O(\epsilon^2), \tag{9.16}$$

where ∂_2 denotes the differentiation with respect to the second variable. The second and the third terms must cancel each other, since the original problem does not depend on τ. That is, the procedure to remove the secular term by setting $\tau = t$ is always correct.

9.9 RG equation as envelope equations

To construct an envelop is a renormalization procedure.[117] Suppose a family of curves $\{x = F(t, \alpha)\}$ parameterized with α is given. Its envelop is given by

$$x = F(t, \alpha), \quad \frac{\partial}{\partial \alpha} F(t, \alpha) = 0. \tag{9.17}$$

The second equation can be interpreted as a renormalization group equation. The envelop curve is such a set of points among the points $\{x, t\}$ satisfying $x = F(t, \alpha)$ that stay invariant under change of α to $\alpha + \delta\alpha$. Therefore, it must satisfy the second equation describing the condition that the (x, t) relation does not change under perturbation of α. This is exactly the same idea as searching features that stay invariant even if microscopic details are perturbed.

However, renormalization group theory must not be misunderstood[118] as a mere theory of special envelop curves. The theory of (or the procedure to make) envelop curves is meaningful only after a one-parameter family of curves is supplied. Thus, from the envelop point of view, the most crucial point is that renormalization provides a principle to construct the one parameter family to which the theory of envelop may be applied. Needless to say, the key to singular perturbation is this principle and not the envelop interpretation, which may not always be useful.

9.10 What the simple example suggests

The above simple example suggests the following:

[117] This was pointed out by T. Kunihiro. T. Wall seems to be the first to construct a result that can be obtained naturally by a renormalization group method as an envelop of approximate solutions: F. T. Wall, "Theory of random walks with limited order of non-self-intersections used to simulate macromolecules," J. Chem. Phys. **63**, 3713 (1975); F. T. Wall and W. A. Seitz, "The excluded volume effect for self-avoiding random walks," J. Chem. Phys. **70**, 1860 (1979). Later, the same method (called the coherent anomaly method) was systematically and extensively used by M. Suzuki to study critical phenomena.

[118] As Kunihiro did

(1) The secular term is a divergence, and renormalization procedure removes this divergence to give the same result singular perturbation methods give. Singular perturbation methods are 'renormalized ordinary perturbations.'[119]
(2) The renormalization group equation is an equation governing slow phenomena. The core of singular perturbation theories is to extract such a slow motion equation, which can be obtained in a unified fashion with the aid of renormalization.[120]

9.11 Resonance due to perturbation

The second class of singular perturbation is due to divergence caused by resonance. If a harmonic oscillator is perturbed by an external perturbation with the frequency identical to the oscillator itself, its amplitude increases indefinitely. For a globally stable nonlinear system, this divergence is checked sooner or later by some nonlinear effect, so there is no genuine divergence. However, if the nonlinear term is treated as perturbation, this effect disappears from the perturbation equations, so singularity due to resonance shows up. Even if the average external force is zero, resonance has a 'secular effect.' That is, there is an effect that accumulates with time. The etymology of 'secular term' lies here.

9.12 Weakly nonlinear oscillators

A typical example illustrating that an ordinary perturbation series is plagued by resonance is the following weak nonlinear oscillator:

$$\frac{d^2y}{dt^2} + y = \epsilon(1 - y^2)\frac{dy}{dt}, \tag{9.18}$$

where ϵ is a small positive constant (so the nonlinearity is weak). This equation is a famous equation called the van der Pol (1889-1959) equation. Introducing the following expansion

$$y = y_0 + \epsilon y_1 + O(\epsilon^2) \tag{9.19}$$

into (9.18), and equating terms with the same power in ε, we obtain

$$\frac{d^2y_0}{dt^2} + y_0 \;\; = \;\; 0, \tag{9.20}$$

[119] As long as the lecturer has experienced, many (almost all?) problems solved by named singular perturbation methods can be solved by renormalization method in a unified fashion without any particular prior knowledge.

[120] Many (all?) famous equations governing phenomenological behaviors (e.g., the nonlinear Schrödinger equation, the Burgers equation, the Boltzmann equation, etc.) can be derived as renormalization group equations.

$$\frac{d^2y_1}{dt^2} + y_1 = (1 - y_0^2)\frac{dy_0}{dt}, \tag{9.21}$$

etc.

9.13 Appearance of secular terms

The general solution to the first equation (9.20) may be written as

$$y_0(t) = Ae^{it} + \text{ c.c.}, \tag{9.22}$$

where A is a complex constant and c.c. implies complex conjugate. Using this in the second equation (9.21), we get

$$\frac{d^2y_1}{dt^2} + y_1 = iA(1 - |A|^2)e^{it} - iA^3e^{3it} + \text{ c.c.} \tag{9.23}$$

We have only to obtain its special solution. To this end it is the easiest to use Lagrange's method of varying coefficients.[121] Thus, we obtain

$$y_1 = \frac{1}{2}A(1 - |A|^2)te^{it} + \frac{i}{8}A^3e^{3it} + \text{ c.c.} \tag{9.24}$$

We have obtained the naive perturbation series as

$$y(t) = Ae^{it} + \epsilon\left[\frac{1}{2}A(1 - |A|^2)te^{it} + \frac{i}{8}A^3e^{3it}\right] + \text{ c.c.} + O(\varepsilon^2). \tag{9.25}$$

Clearly, there is a secular term. The reason for it is that the right-hand side of the equation for y_1 contains the term proportional to e^{it} that has the same frequency as the harmonic oscillator expressed by the left-hand side.

[121] For example, a special solution u to the second order ordinary differential equation

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f$$

may be constructed as follows in terms of the fundamental solutions of the corresponding homogeneous equation ϕ_1 and ϕ_2:

$$u = C_1\phi_1 + C_2\phi_2,$$

where the coefficients (functions) C_1 and C_2 are obtained by solving the following equations:

$$\frac{dC_1}{dx} = -\frac{f\phi_2}{W}, \quad \frac{dC_2}{dx} = \frac{f\phi_1}{W}.$$

Here, W is the Wronskian $W = \phi_1\phi_2' - \phi_2\phi_1'$.

9.14 Renormalization of resonance

Again, numerous singular perturbation methods have been developed to cure secular terms, but our procedure is exactly the same as in the simple example above **9.5**. We separate t as $(t-\tau)+\tau$, and then absorb τ into the constant A that depends on the initial condition. The renormalized result is

$$y(t) = A(\tau)e^{it} + \epsilon\left[\frac{1}{2}A(\tau)(1-|A(\tau)|^2)(t-\tau)e^{it} + \frac{i}{8}A(\tau)^3e^{3it}\right] + \text{ c.c.} + O(\epsilon^2). \quad (9.26)$$

Since y cannot depend on τ, the renormalization group equation $\partial y/\partial \tau = 0$ becomes

$$\frac{dA}{dt} = \epsilon\frac{1}{2}A(1-|A|^2) + O(\epsilon^2), \quad (9.27)$$

where τ is already replaced with t. This is an equation governing the long time behavior of the amplitude. Setting $t = \tau$ in (9.26), we get

$$y(t) = A(t)e^{it} + \epsilon\frac{i}{8}A(t)^3e^{3it} + \text{ c.c.} + O(\epsilon^2). \quad (9.28)$$

The key result is the amplitude equation (9.27).

In this example, the case $\epsilon = 0$ and the case $\epsilon > 0$ are qualitatively different. For a harmonic oscillator, any amplitude is allowed. In contrast, too large or too small amplitudes are not stable for (9.18) as can be seen from its right-hand side term: if $(1-y^2) < 0$, then it is an acceleration term that reduces the amplitude; otherwise, it is an acceleration term injecting energy to the oscillator. Indeed, according to the above approximate calculation, the first order solution slowly converges to a limit cycle expressed by $|A| = 1$. Pragmatically, much simpler calculational method called proto-RG approach exists. See **9.15**

Notice that the amplitude equation (9.27) contains ϵ. As expected from the result of the preceding section, the renormalization group equation describes a slow change of the amplitude (the actual motion is a busy rotation of period about 2π). As already suggested in the preceding section, the renormalization group approach supplies a new point of view for singular perturbation: to extract such a slow motion equation is the key point of the singular perturbation problems.

9.15 Proto RG equation

Let us consider an autonomous equation

$$Ly = \varepsilon N(y), \quad (9.29)$$

where L is a linear operator and N is a nonlinear operator. We make a formal expansion as

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots . \tag{9.30}$$

We know these terms generally contain secular terms. Let us renormalize y_0: according to the spectrum of L, we may write

$$y_0 = \sum_i A_i e_i(t). \tag{9.31}$$

Each y_k has its own secular term Y_k: let us write $y_k = \eta_k + Y_k$. Let us dissect Y_k as

$$Y_k = t \sum_i P_k^{(i)}(A) e_i(t) + Q_k(t, A). \tag{9.32}$$

Thus, we can write y generally as follows:

$$y(t) = \sum_i A_i e_i(t) + t \sum_i P_i(A) e_i(t) + Q(t, A) + R(t, A), \tag{9.33}$$

where $Q(t, A) = \sum_k \varepsilon^k Q_k(t, A)$ is the secular term containing higher powers of t, and R is the rest.

After renormalization (9.33) reads

$$y(t, \tau) = \sum_i A_{Ri}(\tau) e_i(t) + R(t, A_R(\tau)). \tag{9.34}$$

Here, note that $y(t, t) = y(t)$. We have

$$A_{Ri}(\tau) = A_I + \tau P_i(A). \tag{9.35}$$

Let us introduce $\tilde{L}_i$ by

$$L(f(t) e_i(t)) = (\tilde{L}_i f(t)) e_i. \tag{9.36}$$

Then, (9.35) reads

$$\tilde{L}_i A_{Ri} = \tilde{L}_i P_i(A_R). \tag{9.37}$$

This is the proto-RG equation. Notice that the RHS is just $(-)$ the perturbation term containing e_i, so we can read it off from the perturbation equation without solving it.

9.16 Proto RG applied to van der Pol

As an illustration, let us go back to the van der Pol equation (9.18). We have e^{it} and its conjugate as $e_i(t)$. Thus, (9.21) tells us

$$-\tilde{L}_i P_i(A) = iA(1 - |A|^2). \tag{9.38}$$

From (9.36) we get $\mathcal{L}_i$ for this case is

$$\left[\frac{d^2}{dt^2} + 1\right] f(t)e^{it} = -f(t)e^{it} + 2ie^{it}\frac{d}{dt}f(t) + e^{it}\frac{d^2}{dt}f(t) + f(t)e^{it}, \tag{9.39}$$

so we have

$$\mathcal{L}_i = \frac{d^2}{dt^2} + 2i\frac{d}{dt}. \tag{9.40}$$

Therefore, the protoRG equation reads

$$\left[\frac{d^2}{dt^2} + 2i\frac{d}{dt}\right] A_R(t) = \varepsilon i A_R(1 - |A_R|^2), \tag{9.41}$$

but differentiation wrt t gives $O[\varepsilon]$ quantity, so we may keep only the first derivative. Thus, we have obtained the lowest order amplitude equation (9.27) almost for free.

9.17 How reliable is the renormalization group method?

It is easy to prove that (9.28) stays with the true solution within the error of $O[\varepsilon]$ for the time scale $1/\varepsilon$ by a standard argument with the aid of the Grönwall inequality **3.22**. However, such a result never tells us anything definite about the long-term behavior of the system. For example, even the existence of a limit cycle cannot be demonstrated.

A recent work by H. Chiba[122] considerably clarified this problem. Apart from some technicality, his conclusion is: "the long-time behavior of the system with small ε can be qualitatively inferred from its renormalization group equation." That is, roughly speaking, the invariant manifold of the renormalization group equation is diffeomorphic to that of the original equation. In the resonance example above, it is trivial that the renormalization group equation has a hyperbolic limit cycle, so we can conclude that the original equation also has a hyperbolic limit cycle. At least an intuitive explanation of the qualitative reliability of the RG results is attempted in the following.

[122] H. Chiba, "C^1 approximation of vector fields based on the renormalization group method," SIAM J. Appl. Dyn. Syst. **7**, 895 (2008).

9.18 Chiba's logic[123]

(1) The invariant manifold of the renormalization group equation and that of the equation governing the (truncated) renormalized perturbation series are (crudely put; see **9.19**) diffeomorphic.
(2) The differential equation governing the (truncated) renormalized perturbation series is at least C^1-close to the original differential equation.
With (2) and Fenichel's theorem (see **9.20**),
(3) The invariant manifold of the equation governing the (truncated) renormalized perturbation series is diffeomorphic to the invariant manifold of the original equation.

We may expect that the invariant manifolds of C^1-close vector fields are 'close' in some sense. This is, however, a bit delicate question even under hyperbolicity if we demand C^1-closeness of the manifolds.

At least in the case of diffeomorphisms it is known that normal hyperbolicity (see below) is a necessary and sufficient condition for an invariant manifold to persist.[124] Thus, hyperbolicity of invariant manifolds should not be enough to guarantee the qualitative similarity of the renormalization group equation to the original equation.

What Chiba demonstrated is that if the original system has a normally hyperbolic invariant manifold, then the renormalization group equation preserves it. Although for continuous dynamical systems, the relation between the normal hyperbolicity and C^1-structural stability seems not known, probably, we can conjecture that if the original equation is C^1-structurally stable (at least near its invariant manifold), then its renormalization group equation preserves the invariant manifolds of the original system.

9.19 Some technical comments as to Chiba's theory

The relation between the solution $A_R(t)$ to the renormalization group equation and the renormalized perturbation series solution $y(t, 0, A_R(t))$ of the original equation is given by (here maximally the same notations are used as in Section 3.7) the

[123] The explanation may oversimplify and may not do justice to the original theory, so those who are seriously interested in the proof should read the original paper.

[124] M. Hirsch, C. Pugh and M. Shub, "Invariant manifolds," Bull. Amer. Math. Soc. **76**, 1015 (1970), and R. Mañé, "PERSISTENT MANIFOLDS ARE NORMALLY HYPERBOLIC," Trans. Amer. Math. Soc. **246**, 271 (1978). These papers discuss the C^1-closeness of the invariant manifolds.

map α_t defined as $\alpha_t(A) = \sum A_i e_i(t) + \eta(t, A)$ (thus $\alpha_t(A_R(t)) = y(t, 0, A_R(t))$; see (**??**)). Here, we are interested in truncated solutions to some power of ε. Thus, we make a truncated version of α_t by truncating η. The differential equation governing $\alpha_t(A_R(t))$ (both α_t and $A_R(t)$ are truncated) is the equation V' governing the truncated renormalized perturbation series. A technical complication is that the truncated $\alpha_t(A_R(t))$ is generally explicitly time-dependent ($\alpha_t(x)$ is t-dependent), so the invariant set of V' must be considered in the 'space-time' (i.e., (t, y)-space); what (1) asserts is that the invariant set of the truncated original equation V' considered in the (t, y)-space and the direct product of time and the invariant set of the renormalization group equation are diffeomorphic.

(2) should not be a surprise to physicists. The difference between the equation V' governing $y(t, 0, A_R(t))$ (appropriately truncated) and the original equation V must be bounded, since renormalized result is bounded uniformly. Thus, their closeness should be obvious. Here, we need the closeness of the derivatives as well. The approximate solution and the exact solution are differentiable (actually C^1). Therefore, derivatives needed to calculate the derivatives of the vector fields are all bounded (if the appropriate derivatives of formula with respect to y exist). Thus, the equation governing $y(t, 0, A_R(t))$ (appropriately truncated) and the original equation are C^1-close.

The final step is (3); since the equation V' governing $y(t, 0, A_R(t))$ (appropriately truncated) and the original equation V are C^1-close, if normal hyperbolicity (see the next note) of the invariant manifold may be assumed, then Fenichel's theorem concludes the demonstration, if both V and V' are autonomous, but the equation V' governing $y(t, 0, A_R(t))$ (appropriately truncated) is not generally autonomous. Chiba overcame this problem as in (1); to consider the systems in space-time. Fenichel's theorem can be applied there, and the C^1-closeness of the invariant sets is established.

Thus, the invariant manifolds of the renormalization group equation and of the original equation are diffeomorphic.

9.20 Normal hyperbolicity and Fenichel's theorem

Let us consider a (continuous time) dynamical system defined on a subset U of a vector space with a (compact) invariant manifold M. M is assumed to have its unstable and stable manifolds, and the tangent space is decomposed as $T_M U = TM \oplus E^s \oplus E^u$, where E^s is the stable bundle and E^u the unstable bundle. M is normal hyperbolic, if the flow along the manifold M is 'slower' than the flows in the stable and unstable manifolds. The illustration (Fig. 9.1) is basically the same in Fenichel's original paper.

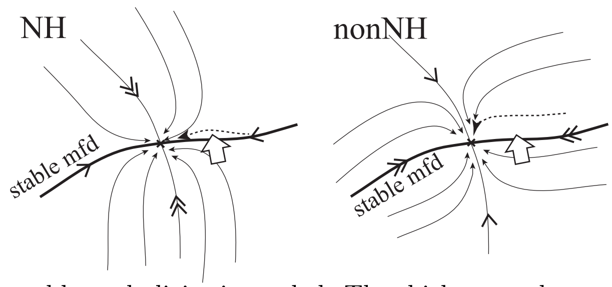

Figure 9.1: Why normal hyperbolicity is needed. The thick curve denotes the invariant manifold and its stable manifold is illustrated. The big white arrow denotes perturbation. 'NH' denotes the normal hyperbolic case, where the flow on the stable manifold is 'faster' (double arrows) than the flow on the invariant manifold. That is, normal direction is 'more' hyperbolic. 'nonNH' denotes the hyperbolic but not normally hyperbolic case. In the NH case the perturbation cannot disrupt the smoothness of the invariant manifold near 'x,' but that is not the case for the nonNH case; a cusp could be formed.

Theorem 3.8.1 [Fenichel][125] Let X be a C^r-vector field ($r \geq 1$) on $\mathbb{R}^n$. Let M be a normally hyperbolic invariant manifold of X without boundary. Then, for any C^r-vector field Y in a certain C^1-neighborhood of X is a Y-invariant manifold C^r-diffeomorphic to M. This diffeomorphism is C^1-close to the identity.

[125] N. Fenichel, "Persistence and smoothness of invariant manifolds for flows," Indiana Univ. Math. J., **21**, 193 (1971).