8 Lecture 8: Bifurcation 2, maps

8.1 Singularities of maps

Consider $f \in C^r(M)$. $x \in M$ such that $f(x) = x$ is called a fixed point of f, which corresponds to a singularity in a vector field: no change in time. We can Taylor expand f around it (often we choose the coordinate system so that $x = 0$) as

$$f(x) = Ax + f_2 + \cdots + f_r + \cdots, \tag{8.1}$$

where f_r is a homogeneous polynomial of degree r. If A has no eigenvalue on the unit circle, the fixed point is called a hyperbolic fixed point. It is structurally stable.

We can introduce all the analogues of normal forms and unfoldings.

8.2 Representative hyperbolic fixed points

According to the eigenvalues of A, we can classify the fixed points. The 2-dimensional case is in Fig. 8.1.

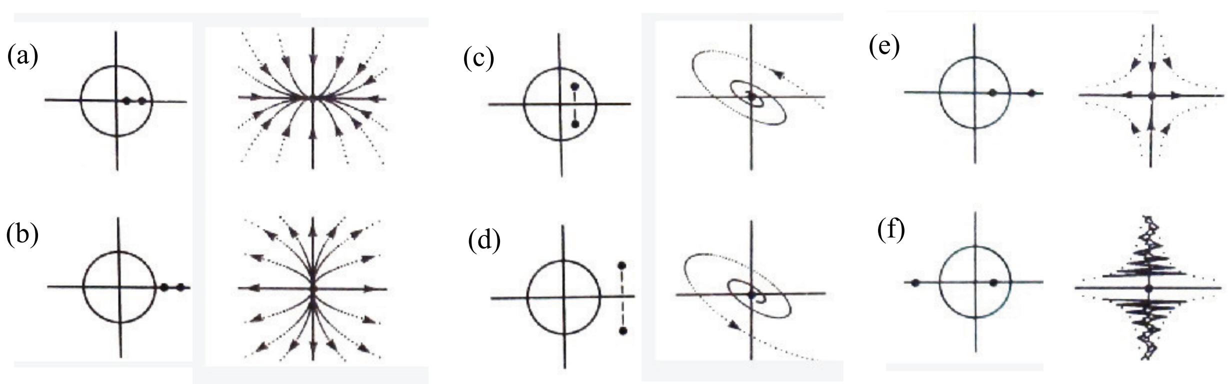

Figure 8.1: Representative hyperbolic fixed points [Fig. 2.1 of AP]

In the rest of this lecture, we discuss an interval map $f : [0,1] \to [0,1]$. That is an endomorphism of $[0,1]$.

8.3 Nonhyperbolic interval endomorphism

At a fixed point if $|f'| = 1$ f is non-hyperbolic, and structurally unstable.

8.4 Versal unfolding of critical map with slope 1

We must distinguish two cases: the fixed point for cases with near the critical parameter value (1) depends on the parameter or (2) not.
Case (1): The normal form with the linear term x generally reads x plus a higher degree polynomial. Thus, $-x + x^2$ is locally (i.e., around the origin) enough. Then, unfolding should be

$$f(\mu, x) = \nu + (1 + \mu)x + Ax^2. \tag{8.2}$$

However, we can scale x and reparameterize this as

$$f(\mu, x) = \nu + x \pm x^2. \tag{8.3}$$

Here, $\pm$ may be replaced by $+$ with an appropriate sign change of ν so we have only to consider $f(y) = \nu + y + y^2$. The bifurcation diagram looks like Fig. 8.2.

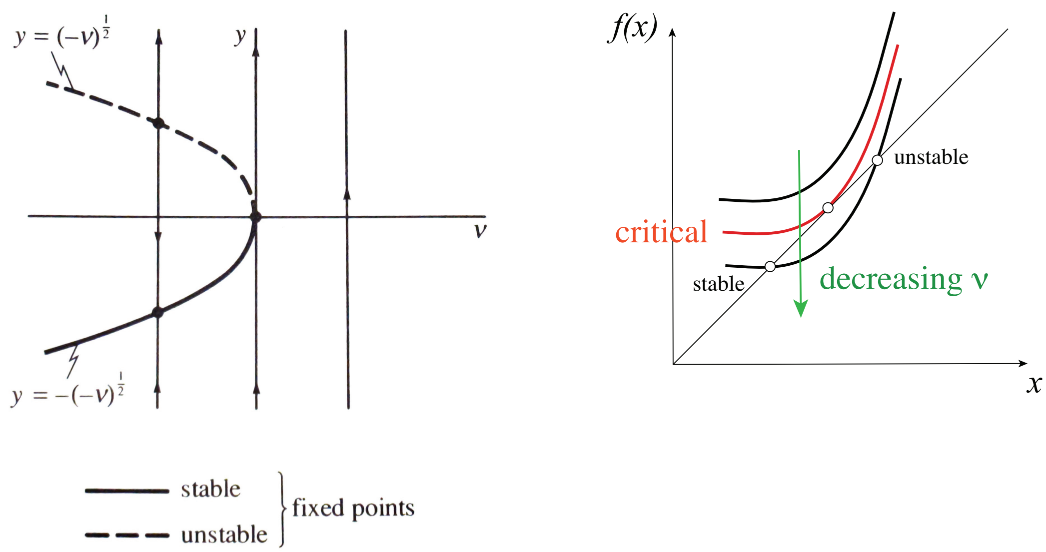

Figure 8.2: Fold: $\nu + y + y^2$; the right is a interval map example indicating a fold bifurcation [Left: Fig. 4.16b of AP]

For case (2): Since we cannot add a constant term, the versal unfolding reads

$$f(\mu, x) = (1 + \mu)x \pm x^2. \tag{8.4}$$

In this case the lowest order term that changes the local topology (a new fixed point) is $\pm x^2$.

The bifurcation diagram reads as Fig 8.3:

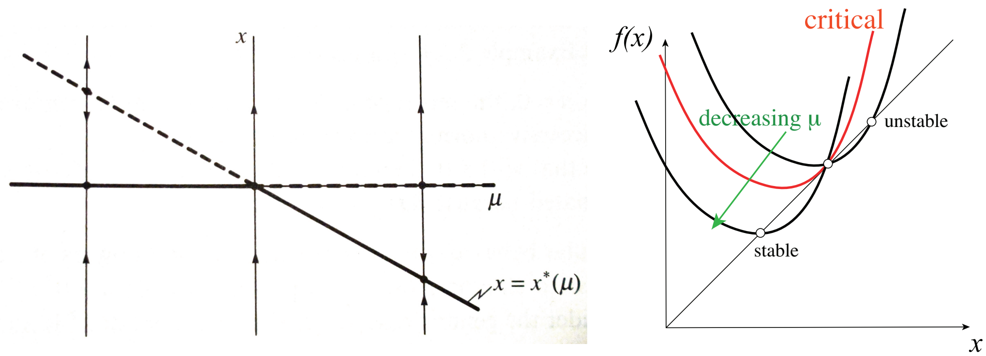

Figure 8.3: Bifurcation diagram for $(1+\mu)x + x^2$; Right. An interval map example.

8.5 Versal unfolding of critical map with slope −1

The case with the slope -1 is a bit complicated, because locally it seems that f does not change much, but f^2 can change qualitatively. Notice that for $f(x) = -x$ $f^2(x) = x$, so it has a fixed line.

Adding $\pm x^2$ (and all other even powers) does not change the topology: suppose $f = -x + x^2$. Then

$$f^2(x) = -(-x + x^2) + (-x + x^2)^2 = x - 2x^3 + x^4 = x(1 + 2x^2 + x^3) \tag{8.5}$$

so no new local fixed point appears.

Consider $f = -x \pm x^3$. Then,

$$f^2(x) = -(-x \pm x^3) \pm (-x \pm x^3)^3 = x - \pm 2x^3 + \cdots, \tag{8.6}$$

so with perturbation new fixed points show up (see Fig. 8.4).

Adding other terms does not alter the qualitative picture. Therefore

$$f(\nu, x) = (-1 + \nu)x \pm x^3 \tag{8.7}$$

is a versal unfolding. This describes the so-called pitch-fork bifurcation.

The bifurcation diagram is Fig. 8.5.

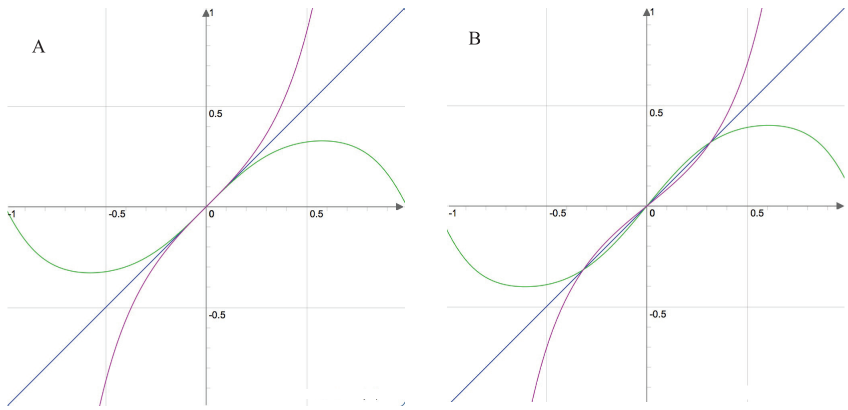

Figure 8.4: f^2 near the slope -1 fixed point. A: at the bifurcation point , B: unfolding. A: Purple: $-x+x^3$, green: $-x-x^3$. B: purple: $-0.9x+x^3$, green: $-1.1x-x^3$

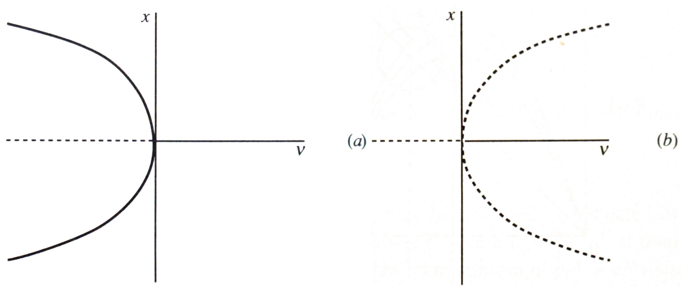

Figure 8.5: pitchfork bifurcation [Fig. 4.18 of AP]

8.6 Pitchfork bifurcations in logistic maps

The logistic map is a map defined on $[0,1]$ as

$$f(x) = ax(1-x). \tag{8.8}$$

Here the parameter $a \in [0,4]$. As a is increased, the map exhibits a series of pitchfork bifurcations as illustrated in the following bifurcation diagram (Fig. 8.6):

These bifurcations correspond to period-doubling bifurcations.

8.7 Feigenbaum critical phenomenon

Initially, Feigenbaum found numerically that
(1) The successive bifurcation parameter value behaves as $a_n = a_\infty - A\delta^{-n}$, where $\delta = 4.66\cdots$.
(2) The pattern size at the n-th bifurcation is compressed as $(-a)^{-n}$, where $a =$

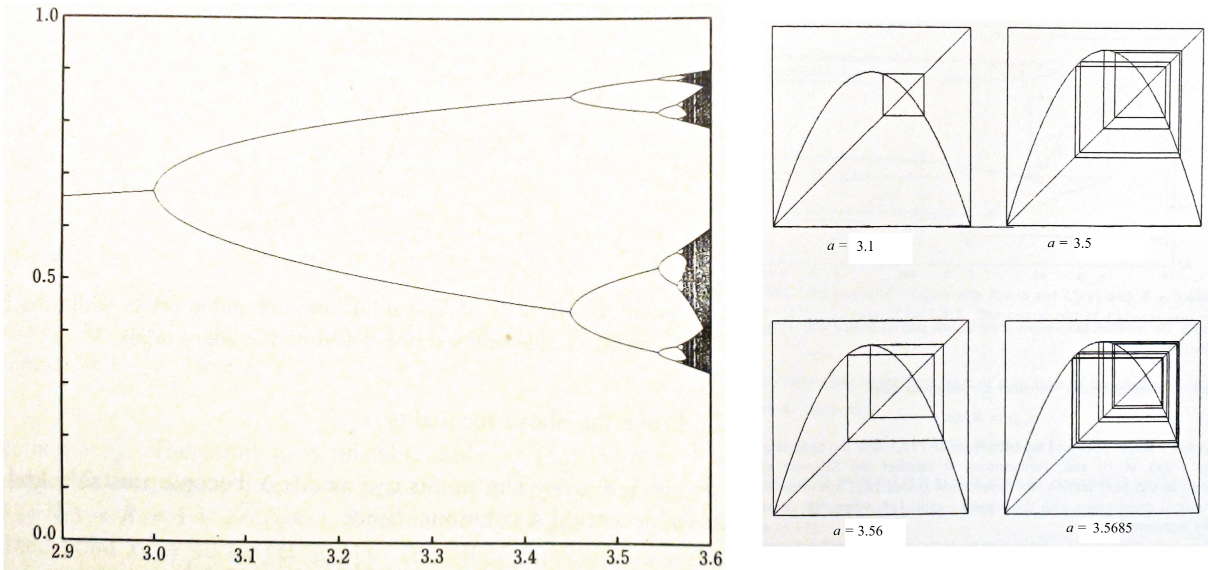

Figure 8.6: Pitchfork accumulating to chaos ($a = 3.5699456\cdots$) in the logistic map [Fig. 3.8. 3.9 of Nagashima & Baba]

$2.5029\cdots$, where $-$ sign implies that the pattern is flipped as n increases by one. He found the universality as well: as long as the map is smooth (C^1), these results do not depend on the map.

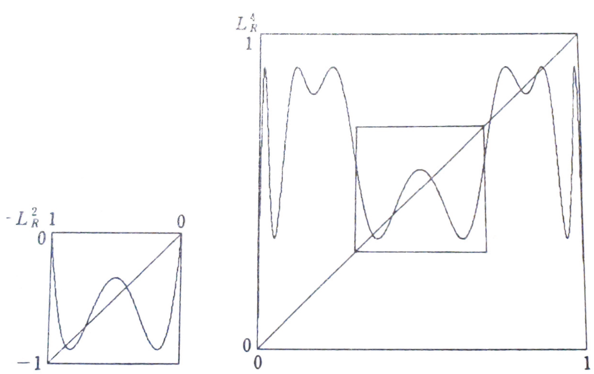

Figure 8.7: f^4 for $a = 3$ is similar to f^4 for $a = 4.44949$ [Fig. 3.11 of Nagashima & Baba]

Let us define

$$g^{[n+1]} = g^{[n]} \circ g^{[n]} \tag{8.9}$$

with $g^{[0]} = g = f^2$.

Notice that we study $g = f \circ f$ or its iterates around $x = 1/2$ (see Fig. 8.7). From the figure we can guess that between the n-th and the $n+1$-pitchfork bifurcations (near $x = 1/2$)

$$g^{[n]}(a, x) = (-\alpha)^{-n} h^{[n]}(\varepsilon_n, y_n) + \frac{1}{2}, \tag{8.10}$$

where

$$y_n = (x - 1/2)(-\alpha)^n, \ \varepsilon_n = A(a - a_\infty)\delta^n. \tag{8.11}$$

Feigenbaum's conjecture is that the following limit is well-defined:

$$h^{[n]}(\varepsilon_n, y_n) \to h(\varepsilon, y) \tag{8.12}$$

for any 'smooth' g.

8.8 Renormalization around a_∞

(8.9) in terms of $h^{[n]}$ can be obtained as follows (note that $y_n = h^{[n]}(\varepsilon_n, y_n)$):

$$(-\alpha)^{-n-1} h^{[n+1]}(\varepsilon_{n+1}, y_{n+1}) + \frac{1}{2} = (-\alpha)^{-n} h^{[n]}(\varepsilon_n, h^{[n]}(\varepsilon_n, y_n)) + \frac{1}{2}. \tag{8.13}$$

Therefore,

$$(-\alpha)^{-(n+1)} h^{[n+1]}(\varepsilon_{n+1}, y_{n+1}) = (-\alpha)^{-n} h^{[n]}(\varepsilon_n, h^{[n]}(\varepsilon_n, y_n)) \tag{8.14}$$

or

$$(-\alpha)^{-1} h^{[n+1]}(\delta\varepsilon_n, -\alpha y_n) = h^{[n]}(\varepsilon_n, h^{[n]}(\varepsilon_n, y_n)). \tag{8.15}$$

Therefore, h satisfies

$$-\alpha^{-1} h(\delta\varepsilon, -\alpha y) = h(\varepsilon, h(\varepsilon, y)). \tag{8.16}$$

This is the RG fixed point equation.

8.9 Approximate solution of the RG equation

(8.16) is not easy to solve. Therefore, we use the following Ansatz: $\varepsilon = 0$ is the critical point, so

$$h(0, y) = 1 - cy^2. \tag{8.17}$$

For this to satisfy

$$-\alpha^{-1} h(0, -\alpha y) = h(0, h(0, y)) \tag{8.18}$$

we have

$$-\alpha^{-1}(1 - c(\alpha y)^2) = 1 - c(1 - cy^2)^2 = 1 - c + 2c^2y^2 - c^3y^4 \tag{8.19}$$

Truncating this at $O[y^2]$, we require

$$-\alpha^{-1} = 1 - c \tag{8.20}$$

$$c\alpha = 2c^2. \tag{8.21}$$

We can determine $\alpha = 1 + \sqrt{3} = 2.732$. The empirical value is 2.5029. Not too bad.

8.10 Determination of δ

To determine δ we mus point function to study the $\varepsilon \neq 0$ case, so we study the deviation ψ_n from the fixed point.

$$h^{[n]}(\varepsilon_n, y_n) = h(0, y_n) + \varepsilon_n \psi_n(y_n). \tag{8.22}$$

We introduce this into (8.15) and linearize as

$$(-\alpha)^{-1}[h(0, -\alpha y_n) + \varepsilon_{n+1}\psi_n(-\alpha y_n)] = h(0, h(0, y_n)) + \varepsilon_n \partial_y h(0, h(0, y_n))\psi_n(y_n) + \varepsilon_n \psi_n(h(0, y_n)). \tag{8.23}$$

Define $\mathcal{G}$ as

$$\mathcal{G}\phi(y) = -\alpha[\partial_y h(0, h(0, -y/\alpha))\phi(-y/\alpha) + \phi(h(0, -y/\alpha))] \tag{8.24}$$

Since $(-\alpha)^{-1}h(0, -\alpha y_n) = h(0, h(0, y_n))$, we get the following equation ($y_{n+1} = -\alpha y_n$)

$$\varepsilon_{n+1}\psi_n(y) = \varepsilon_n \mathcal{G}\psi_n(y), \tag{8.25}$$

That is,

$$\delta\psi_n(y) = \mathcal{G}\psi_n(y). \tag{8.26}$$

The eigenvalue of $\mathcal{G}$ gives δ. We know $h(0, y) = 1 - (1 + \alpha^{-1})y^2$ so we can compute $\mathcal{G}$. However, I could not get a good approximate value for $\delta = 4.66\cdots$. Try.