7 Lecture 7: Bifurcation of vector fields

7.1 Family of dynamical systems and bifurcation

Let $\{X_\mu \in \mathcal{X}^r(M)\}_{\mu \in B}$ be a family of C^r vector field on M, where B is a set of parameter values (can be a set of vectors). We say this family exhibits a bifurcation at $\mu^* \in B$, if every neighborhood of μ^* contains μ such that X_{μ^*} and X_μ are topologically distinct (i.e., there is no (local) homeomorphism between them; intuitively speaking, qualitatively distinct). We may say X_{μ^*} is not structurally stable.

We have already encountered the Hopf bifurcation.

7.2 What are the key questions about bifurcations?

Suppose we have a vector field $X \in \mathcal{X}^r(M)$ which is not structurally stable. The most interesting question must be: what happens if we modify X a bit in $\mathcal{X}^r(M)$? Needless to say, we are not interested in reparametrization (or chart change) of the vector field, so we wish to classify the fields near X modulo homeomorphism of the fields and reparametrization of the 'bifurcation parameters' that describe deviation of the fields from the original X.

7.3 Versal unfolding

Thus, we wish to make a family $\{X_\mu\}$ with $X_0 = X$ that is most general and the 'simplest.' Such a family is called the versal unfolding of X.

Here, any X_μ such that $X_0 = X$ is called an unfolding of X.

An unfolding is the versal unfolding, if any other unfolding is equivalent to it. Here 'equivalence' means the same modulo homeo and reparametrization of the parameters.

To construct the versal unfolding, we use two steps: reduction of vector fields to the normal form, and a much more subtle mathematics suh as Malgrange's preparation theorem.

7.4 Malgrange's preparation theorem

Let $F(\mu, x) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a smooth function defined on a neighborhood of the origin of $\mathbb{R}^n \times \mathbb{R}$. Here we discuss the simplest cas where the dynamics is in 1 space (but the parameter is n-vector).

Theorem.[92] Suppose $F(0,x) = x^k g(x)$, where g is smooth in a neighborhood of $x = 0$ and $g(0) \neq 0$. Then, there is a smooth function $q(\mu, x)$ in a neighborhood of $(\mu, x) = (0,0)$, and functions $s_i(\mu)$ ($i \in \{0, 1, \cdots, k-1\}$) smooth in a neighborhood of $\mu = 0$ such that

$$q(\mu, x)F(\mu, x) = x^k + \sum_{i=0}^{k-1} s_i(\mu)x^i. \tag{7.1}$$

7.5 Versal unfolding of $-x^2$

Let us consider $X = -x^2$. Certainly, this is structurally unstable. The above theorem tells us that

$$X_{a,b} = -x^2 + 2ax + b \tag{7.2}$$

is a versal unfolding. However, this can be rewritten as $-(x-a)^2 + b + a^2$, so a homeo can change this to $-x^2 + \mu$ form. Therefore, we can conclude that the versal unfolding is $X_\mu = -x^2 + \mu$. We will use this result later.

A versal unfolding of $\dot{x} = -x^3$ is $\dot{x} = \nu_1 x + \nu_2 x^2 - x^3$.

7.6 Normal form

Nere $x = 0$ formally a vector field has the following form

$$X = Ax + \sum_{r \geq 2} X^r(x), \tag{7.3}$$

where A is the derivative at $x = 0$ and where X^r is an order r polynomial vector field Since we are interested in a bifurcation at the bifurcation point usually A has a special feature, say, some 0 eigenvalues. If we look at Malgrange type theorems **7.4**, we see that the versal family is determined by the lowest nontrivial order X^r at least locally. Thus, it is very convenient to eliminate lower order polynomial terms as much as possible by a coordinate transformation $x \to y$ so that

$$X = Ay + \sum_{r \geq s} X^r(y), \tag{7.4}$$

as much large s as possible. This form is called the normal form of X.

The conversion of the original X to its normal form is done order by order.

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[92] Needless to say there is a version for higher-dimensional spaces. This is a key theorem for catastrophe theory, but not easy to prove.

7.7 Formal elimination of degree r term

Consider

$$\dot{x} = X(x) = Ax + X^r(x) + O[|x|^{r+1}], \tag{7.5}$$

where A is $dA/dx|_{x=0}$, and X^r is a sum of x^k with $|k| = r$ (degree r polynomial term).

Introduce

$$x = y + h^r(y), \tag{7.6}$$

where h^r is a degree r polynomial. Using this, we try to eliminate X^r from (7.5).

(7.6) may be inverted as

$$y = x - h^r(x) + O[|x|^{r+1}]. \tag{7.7}$$

Thus, $(d/dx = D)$

$$\begin{aligned}
\dot{y} &= \dot{x} - Dh^r(x)\dot{x} + O[|x|^r]\dot{x} && (7.8)\\
&= Ax + X^r(x) - Dh^r(x)Ax + O[|x|^r]\dot{x} && (7.9)\\
&= A[y + h^r(y)] + X^r(y) - Dh^r(y)Ay + O[|y|^{r+1}] && (7.10)\\
&= Ay - [Dh^r(y)Ay - Ah^r(y)] + X^r(y) + O[|y|^{r+1}] && (7.11)
\end{aligned}$$

Notice that (with the summation convention)

$$(Dh^r(y)Ay)_k = D_i h^r_k(y) A_{ij} y_j = \frac{\partial h^r_k}{\partial y_i} A_{ij} y_j = \left(A_{ij} y_j \frac{\partial}{\partial y_i} \right) h^r_k \tag{7.12}$$

and

$$(Ah^r(y))_k = A_{kj} h^r_j \tag{7.13}$$

Introduce a linear operator L_A (called the Lie bracket) as[93]

$$Dh^r(y)Ay - Ah^r(y) = \left(A_{ij} y_j \frac{\partial}{\partial y_i} \right) h^r_k - A_{kj} h^r_j \equiv L_A h^r. \tag{7.14}$$

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[93] L_A is the Lie derivative operator. In a more standard language (7.14) reads

$$\left[A_{ij} y_j \frac{\partial}{\partial y_i}, h^r_k \frac{\partial}{\partial y_k} \right] = A_{ij} y_j \frac{\partial h^r_k}{\partial y_i} \frac{\partial}{\partial y_k} - h^r_j A_{kj} \frac{\partial}{\partial y_k} = \left\{ A_{ij} y_j \frac{\partial h^r_k}{\partial y_i} - h^r_j A_{kj} \right\} \frac{\partial}{\partial y_k}.$$

Then, (7.11) reads

$$\dot{y} = Ay - L_A h^r(y) + X^r(y) + O[|y|^{r+1}]. \tag{7.15}$$

If we can solve

$$L_A h^r(y) = X^r(y) \tag{7.16}$$

we are done. This is the non-resonance condition.

7.8 Lie bracket with resonance

When the 'non-resonance condition' is not satisfied, the procedure in **7.7** cannot remove X^r totally. The elements in the cokernel ($= [\mathrm{Im}(L_A)]^c$) of L_A survive.

7.9 Normal form theorem

$X \in \mathcal{X}^r$ with $X(0) = 0$, the differential equation with $DX(0) = 0$

$$\dot{x} = Ax + X(x) \tag{7.17}$$

may be transformed by a polynomial transformation $y = x + h(x)$, where h is a polynomial of second or higher degree, to

$$\dot{y} = Ay + \sum_{r=2}^{N} Y^r + O[|y|^{N+1}]., \tag{7.18}$$

where Y^r is a polynomial of degree r in the cokernel of L_A.

Let us consider the following example:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \sum_{r\geq 2} X^r(x). \tag{7.19}$$

The nonresonance condition (7.12) reads as follows.

$$L_A = \left[2y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2}, \right] \tag{7.20}$$

Let h^r be

$$\sum_{m+n=r} a^1_{mn} y_1^m y_2^n \frac{\partial}{\partial y_1} + \sum_{m+n=r} a^2_{mn} y_1^m y_2^n \frac{\partial}{\partial y_2}. \tag{7.21}$$

Therefore,

$$\begin{aligned}[L_A, h^r] &= 2y_1\frac{\partial}{\partial y_1}\sum_{m+n=r}\left\{a^1_{mn}y_1^m y_2^n\frac{\partial}{\partial y_1} + a^2_{mn}y_1^m y_2^n\frac{\partial}{\partial y_2}\right\} - \sum_{m+n=r}\left\{a^1_{mn}y_1^m y_2^n\frac{\partial}{\partial y_1} + a^2_{mn}y_1^m y_2^n\frac{\partial}{\partial y_2}\right\}2y_1\frac{\partial}{\partial y_1} \\ &\quad + y_2\frac{\partial}{\partial y_2}\sum_{m+n=r}\left\{a^1_{mn}y_1^m y_2^n\frac{\partial}{\partial y_1} + a^2_{mn}y_1^m y_2^n\frac{\partial}{\partial y_2}\right\} - \sum_{m+n=r}\left\{a^1_{mn}y_1^m y_2^n\frac{\partial}{\partial y_1} + a^2_{mn}y_1^m y_2^n\frac{\partial}{\partial y_2}\right\}y_2\frac{\partial}{\partial y_2}\end{aligned} \tag{7.22}$$

From this the y_1 component reads

$$\sum_{m+n=r}(2ma^1_{mn}y_1^m y_2^n + na^1_{mn}y_1^m y_2^n - 2a^1_{mn}y_1^m y_2^n)\frac{\partial}{\partial y_1} = \sum_{m+n=r}a^1_{mn}(2m+n-2)y_1^m y_2^n\frac{\partial}{\partial y_1}, \tag{7.23}$$

and similarly the y_2 component reads

$$\sum_{m+n=r}a^2_{mn}(2m+n-1)y_1^m y_2^n\frac{\partial}{\partial y_2}. \tag{7.24}$$

The terms with vanishing coefficients are resonant terms and we cannot eliminate them. For $r = 2$ for y_1 we must solve $m + n = 2$ and $2m + n = 2$. That is $m = 0$, $n = 2$. For y_2 we must solve $m + n = 2$ and $2m + n = 1$. There is no solution. For larger r $m = 1 - r$ means not solution at all. That is, all the terms can be removed. Thus the lowest order normal form reads

$$\dot{y}_1 = 2y_1 + Ky_2^2, \tag{7.25}$$
$$\dot{y}_2 = y_2. \tag{7.26}$$

7.10 Saddle-node bifurcation

Ler us study the case illustrated in Fig. 4.5(a). In this case for A one eigenvalue maintains its sign, and the second one changes its sign:

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} \tag{7.27}$$

at the bifurcation point. Let us convert the original equation into a normal form. The resonance condition is (a good exercise):
For the first component: $m + n = r$, $m\lambda = \lambda$ (i.e., $m = 1$). $m = n = 1$ for $r = 2$, for

$r \geq 3$ $m = 1$ $n = r - 1$. Thus, $x + Axy + B_r xy^{r-1}$.
For the second component: $m + n = r$, $m = 0$. That is, $\sum_{r \geq 2} a_r y^r$.
In summary, the normal form around the saddle-node bifurcation point is

$$\dot{x} = \lambda x + \sum a_r xy^{r-1}, \tag{7.28}$$

$$\dot{y} = \sum b_r y^r. \tag{7.29}$$

To determine a convenient versal unfolding, we truncate this at $r = 2$

$$\dot{x} = \lambda x + Axy, \tag{7.30}$$

$$\dot{y} = By^2. \tag{7.31}$$

As Malgrange's theorem tells us, adding lower order polynomials to the above normal form gives a versal unfolding. However, as we noted in **7.5** we can eliminate some terms by coordinate transformation

$$\dot{x} = \lambda x + Axy, \tag{7.32}$$

$$\dot{y} = \nu + By^2. \tag{7.33}$$

For the $\lambda > 0$ $B < 0$ case, the bifurcation diagram looks like Fi.g 7.1.

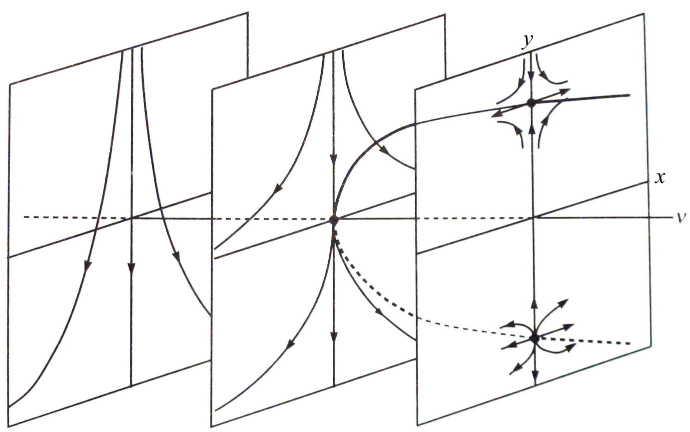

Figure 7.1: Supercritical saddle-node bifurcation [Fig. 4.5 of Arrowsmith and Place p201]

7.11 Hopf singularity

In this case for A conjugate complex eigenvalues change the sign of their real part.

At the bifurcation point

$$A = \begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}. \tag{7.34}$$

In this case after complexification we can still diagonalize A as $A = [i\beta, 0i\beta]$.

7.12 Normal form around Hopf singularity

Let us convert the original equation into a normal form. The resonance condition is For the first component z: $m + n = r$, $im\beta - in\beta = i\beta$ (i.e., $m - n = 1$). Thus, $i\beta z + z\sum B_r(z\overline{z})^r$.
For the second component $\overline{z}$: $m+n = r$, $im\beta - in\beta = -i\beta$ (i.e., $m-n = -1$). Thus, $-i\beta\overline{z} + \overline{z}\sum C_r(z\overline{z})^r$.
Therefore, the normal form reads

$$\dot{z} = i\beta z + z\sum B_r(z\overline{z})^r, \tag{7.35}$$

and its conjugate. In terms of the original variables

$$\begin{aligned} \dot{x} &= -\beta y + \sum(Re(B_r)x - Im(B_r)y)(x^2+y^2)^r, & (7.36) \\ \dot{y} &= \beta x + \sum(Im(B_r)x + Re(B_r)y)(x^2+y^2)^r. & (7.37) \end{aligned}$$

7.13 Versal unfolding of Normal form for Hopf bifurcation

Locally we may truncate (7.37) as

$$\begin{aligned} \dot{x} &= -\beta y + (ax - by)(x^2+y^2) + O[|x|^5], & (7.38) \\ \dot{y} &= \beta x + (bx + ay)(x^2+y^2) + O[|x|^5] & (7.39) \end{aligned}$$

We wish to keep the circular symmetry and mirror symmetry, Then there should not be any even order terms and the unfolding looks like

$$\begin{aligned} \dot{x} &= \nu x - \beta y + (ax - by)(x^2+y^2) + O[|x|^5], & (7.40) \\ \dot{y} &= \beta x + \nu y + (bx + ay)(x^2+y^2) + O[|x|^5] & (7.41) \end{aligned}$$

7.14 Hopf bifurcation diagram

We can plot the local family described by (7.41). If $a > 0$ it is called the subcritical bifurcation; if $a < 0$ it is called the supercritical bifurcation (Fig. 7.2).

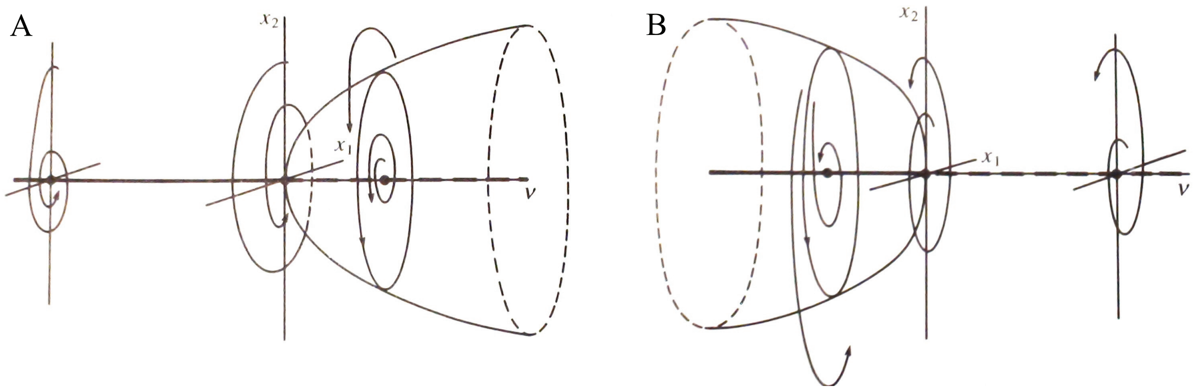

Figure 7.2: A: supercritical Hopf bifurcation, B: subcritical (or inverted) [Fig. 4.7 of AP]