6 Lecture 6: Periodic orbit and limit cycle

6.1 Three types of trajectories

For real ODE $\dot{x} = v(x)$ with a smooth vector field, trajectories are diffeomorphic to a point, a circle (S^1) or a line.

Thus a phase curve of an equation always has a simple intrinsic geometry.[81]

6.2 Periodic orbit (cycle)

A trajectory diffeomorphic to S^1 is called a periodic orbit or a cycle.

6.3 First return map = Poincare map = monodromy transformation

For a smooth vector field, if there is a periodic orbit, we can take a transversal hypersurface (= codimension one surface perpendicular to the orbit; often called a Poincare surface) crossing the orbit at a point p. The orbits starting sufficiently close to p on this surface will return to a neighborhood of p and are again transversal to the surface (Fig. 6.1). Thus, we can locally define a map from the surface into itself. This map[82] is called the first return map, Poincare map or monodromy transformation.

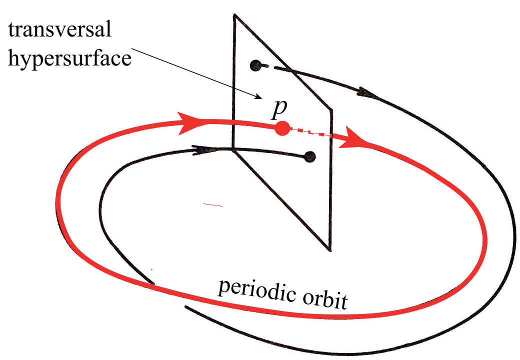

Figure 6.1: First return map

The periodic orbit corresponds to a fixed point of this map.

The first return map does not depend on the choice of the Poincare surface (all

[81] However, complicated knots can be formed as we will see in the Lorenz system.

[82] More precisely, its germ

diffeomorphic locally).

6.4 Linearization around periodic orbit

In a tubular nbh of a periodic orbit γ of period T we can linearize the flow as

$$\dot{x} = DX(\gamma(t))x = A(t)x, \tag{6.1}$$

where $A(t) = DX(\gamma(t))$ is a linear operator with period T: $A(t+T) = A(t)$. Its solution may be written as

$$x(t) = F(t)x_0, \tag{6.2}$$

where $F(t)$ is a linear operator that may be formally written in terms of time ordered exponential of the integral of $DX(\gamma(t))$. F is called a fundamental solution matrix.

$F(T)$ must be the linearization of the Poincare map. It is called the monodromy matrix.

6.5 Floquet's theorem about fundamental matrix for periodic system[83]

Theorem. The fundamental matrix $F(t)$ in **6.4** maybe written as

$$F(t) = B(t)e^{2\pi\Lambda t}, \tag{6.3}$$

where $B(t+T) = B(t)$ and Λ a constant matrix.
[Demo]
Notice that $\Phi(t) = F(t+T)$ is also a fundamental matrix:

$$\frac{d}{dt}\Phi(t) = A(t+T)\Phi(t) = A(t)\Phi(t). \tag{6.4}$$

Since $F(t)$ is invertible for all t, so is $\Phi(t)$. Therefore, Φ is also a fundamental solution matrix. Therefore, there is an invertible matrix C such that $F(t+T) = F(t)C$ for all t (for example $C = F(T)$ is a possible choice). Its log is well defined, so we can introduce Λ as

$$2\pi T\Lambda = \log C. \tag{6.5}$$

Let us write

$$B(t) = F(t)e^{-2\pi\Lambda t}. \tag{6.6}$$

[83] Its 3D version is Bloch's theorem for solid state physics.
'Time crystal' is somewhat related, which is a quantum many-body phenomenon. See Zhang et al., Observation of a discrete time crystal, Nature **543**, 217 (2017) and papers quoted in its introduction; I do not recommend the recent Phys Today article.

Then,

$$B(t+T) = F(t+T)e^{-2\pi\Lambda(t+T)} = F(t)e^{2\pi\Lambda T}e^{-2\pi\Lambda(t+T)} = F(t)e^{-2\pi\Lambda t} = B(t). \quad (6.7)$$

Thus, B is invertible and periodic.

The eigenvalues of $e^{2\pi T\Lambda}$ are called the Floquet multipliers governing the behavior of the Poincaré map, and the eigenvalues ($\times\pi$) of Λ is called the characteristic exponents.[84]

We can restate the theorem as

Theorem There is an invertible periodic linear transformation $B(t)$ such that $x = B(t)y$ transforms the original equation to

$$\dot{y} = \Lambda y. \quad (6.8)$$

Thus, the real part of the eigenvalues of Λ is called a Lyapunov exponent. In terms of Lyapunov exponents, we can discuss asymptotic stability of the periodic orbit.

6.6 Limit cycle

A limit cycle is an isolated phase curve diffeomorphic to a circle. In other words, a closed curve is called a limit cycle if it corresponds to an isolated fixed point of the first return map.

The multiplicity of a limit cycle is the multiplicity of the corresponding fixed point of its return map.

6.7 Stability of limit cycle

A periodic orbit is orbitally stable (Lyapunov stable), if any orbit staring in a certain tubular neighborhood of the orbit stays init.

A periodic orbit is orbitally asymptotically stable. if it is orbitally stable and in the $t \to \infty$ limit any orbit stating from a point in its tubular neighborhood converges to the orbit (its fixed point of the return map is asymptotically stable).[85]

---

[84] In some books, any number μ such that e^{μ} becomes a Floquet multiplier is called a characteristic exponent. In this case its imaginary part is not unique.

[85] ⟪**Stability of biological clock**⟫ Michele Monti, David K. Lubensky, and Pieter Rein ten Wolde, Robustness of Clocks to Input Noise, PRL 121 078101 (2018) "Here, using models of the Kai system of cyanobacteria, we compare a limit-cycle oscillator with two hourglass models, one

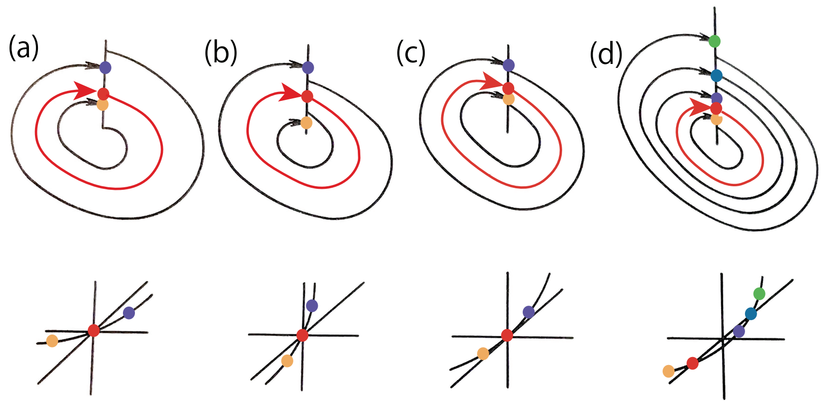

Figure 6.2: Limit cycle on 2-space and its return maps: $I \to I$, where I is an appropriate interval. (a) Stable limit cycle, (b) Unstable limit cycle, (c) semistable limit cycle which is not structurally stable, (d) an outcome of perturbation of (c).

As noted already, we can use Lyapunov exponents for the periodic orbit to study its stability (**6.5**).

6.8 Poincare-Bendixson's theorem

Let D be a subset of S^2.[86] For a flow on D if α and ω-limit sets of $p \in D$ do not contain a singular point, it is a periodic orbit.

A proof is given in **6.9-6.10**. To begin with we need to know that the limit sets of a point is a connected set.
https://www.youtube.com/watch?v=uEfB5DG9x9M&frags=pl%2Cwn illustrates the idea of a proof of the theorem after ca 10 min.

6.9 Limit sets are connected

We consider a flow on a compact manifold M or its subset.

Take $\omega(p)$ and suppose it is not connected. Limit sets are closed sets, so $\omega(p)$ must consist of at least two closed sets ω_1 and ω_2. Since they are closed, they have unoverlapping neighborhoods U_1 and U_2,[87] Since they are ω-limit sets, there must be a sequence of points on the orbit such that $x_i \to x \in \omega_1$ and $y_i \to y \in \omega_2$. We can

---

that without driving relaxes exponentially and one that does so in an oscillatory fashion. In the limit of low input noise, all three systems are equally informative on time, yet in the regime of high input-noise the limit-cycle oscillator is far superior."

[86] or its open subset or P^2.

[87] We assume Hausdorff.

choose these sequences as $x_1 < y_1 < x_2 < y_2 < \cdots$ along the orbit. For sufficiently large n $x_n \in U_1$ and $y_n \in U_2$, so we can choose z_n on the orbit between x_n and y_n but outside $U_1 \cup U_2$. Since M is compact, we can choose a converging subsequence from $\{z_k\}$, but it converges somewhere other than ω_1 nor ω_2, so this sequence misses $\omega(p)$, a contradiction.

6.10 Demonstration of Poincare-Bendixson's theorem

If p is on a periodic orbit, there is nothing to show. Let us assume p is not on a periodic orbit and $\omega(p)$ does not contain any singular point.
(i) Let $p' \in \omega(p)$. Then, the orbit $C(p')$ going through p' is a periodic orbit.
(ii) $C(p') \subset \omega(p)$. If they do not agree, since both must be closed, $\omega(p) \setminus C(p')$ is not a closed set. Since ω must be connected, there must be a point $a \in \omega(p)$ such that $a \in C(p') \cap \overline{\omega(p) \setminus C(p')}$.
(iii) Take a small neighborhood U of a. Since $a \in C(p') \cap \overline{\omega(p) \setminus C(p')}$, there must be $b \in U \cap (\omega(p) \setminus C(p'))$. Since a is not a singular point, we can make a transversal line ℓ through a (see Fig. 6.3). Since $b \in U$, the vector through it must be close to that at a, so the orbit through b crosses ℓ at $c \in U$.
(iv) This $c \notin C(p')$, since if in $C(p')$, so is $b \in C(p')$, contradicting $b \in \omega(p) \setminus C(p')$. Thus, c is recurrent and not periodic.
(v) Since $a \in \omega(p)$, there are $\{a_k\}$ converging to a and on ℓ in this order along the orbit. Take a_1 and a_2. Then the orbit extended beyond a_2 must be in the green shaded region in Fig. 6.3. Thus, a_3 must be between a_2 and a, etc. That is, $\lim a_k = a$, implying $\omega(p) \cap \ell = \{a\}$, unique. This contradicts the existence of c. Therefore, $\omega(p) = C(p') = C(p)$, a periodic orbit.

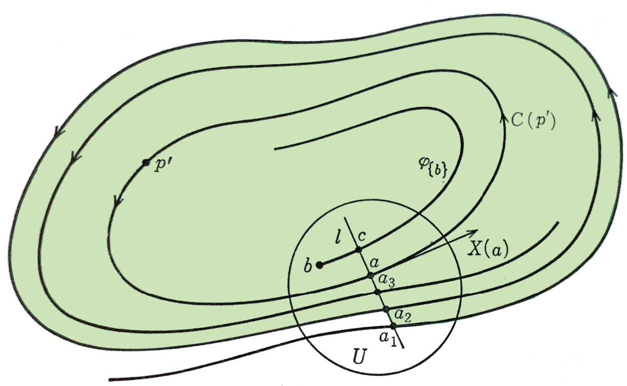

Figure 6.3: [Fig. 3.2 of Tamura, color added]

6.11 Poincare-Bendixson's theorem: a more general version

Let $X \in \mathcal{X}^r(S^2)$ be a vector field with a finite number of singularities. For any $p \in S^2$ one of the followings holds:
(1) $\omega(p)$ is a singularity = fixed point.
(2) $\omega(p)$ is a periodic orbit.
(3) $\omega(p)$ consists of singularities $\{p_i\}$ and regular orbits γ such that if $\gamma \subset \omega(p)$, then $\alpha(p_i)$ and $\omega(p_j)$.

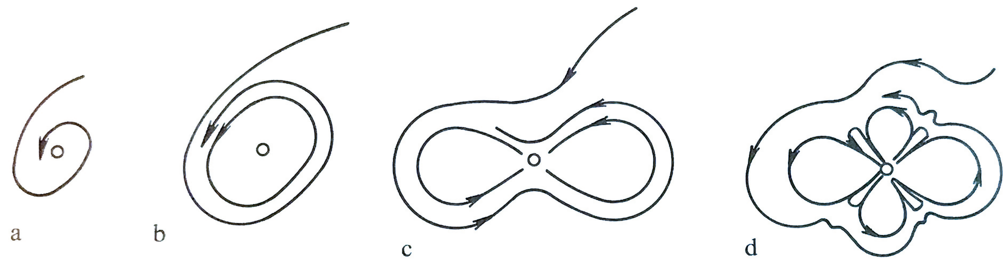

Figure 6.4: Possible $\omega(p)$ [Fig. 7 of DS I]

6.12 Bendixson's criterion for no closed orbit

On $\mathbb{R}^2$, consider $\dot{x} = X(x)$. On a simply connected region $D \subset \mathbb{R}^2$, div$X$ is positive or negative semidefinite, then there is no closed orbit lying entirely in D.
[Demo]
For any closed curve γ Green's theorem tells us

$$\int_\gamma X \times (dx, dy) \neq 0. \tag{6.9}$$

However, if γ is a solution curve, then X must be parallel to (dx, dy), so the integral must vanish.

6.13 How many limit cycles are there?

There are a set of theorems on various kinds of vector fields called the finiteness theorems. For example,
A polynomial vector field on the real plane has only a finite number of limit cycles.[88]

This follows from a much more general theorem:

[88]Theorem 1 on p106 of DS I.

Limit cycles of an analytic vector field on a 2-surface cannot accumulate to a compound cycle of the field.[89].

Hilbert's 16th problem is to prove:
The number of limit cycles of a polynomial vector field (of order n) in the real plane is bounded by a number N depending only on n.

This is still open even for $n = 2$.

6.14 Structurally unstable examples

Although we will pay our main attention to generic cases, we must know simple structurally unstable behaviors as illustrated in Fig. 6.5

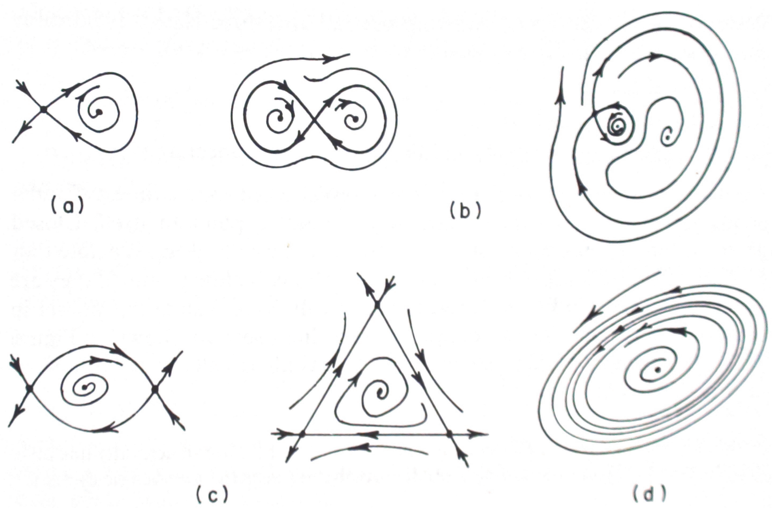

Figure 6.5: Some structurally unstable limit sets (a) homoclinic orbits (saddle loops); (b) Double saddle loop; (c) Homoclinic cycles; (d) periodic orbit band [Fig. 1.81 of GH]

6.15 Chemical oscillation

Belousov discovered the so-called BZ reaction (Belousov-Zhabotinsky reaction). This reaction is basically the oxidation of malonic acid by bromic acid $HBrO_3$. If ferroin is used as a catalyst the oscillation may be observed as a color oscillation of the solution between blue and red.[90]
Stirred: https://www.youtube.com/watch?v=eSWyxMWXw00&frags=pl%2Cwn.

[89] Theorem e of DS I p 106.

[90] Belousov quit science because he could not publish his fundamental work on this reaction

Not-stirred: https://www.youtube.com/watch?v=IBa4kgXI4Cg&frags=wn

The reaction involves numerous (likely to be > 50) chemical species, but a simplified model based on actual observations was proposed as the Oregonator model:[91]

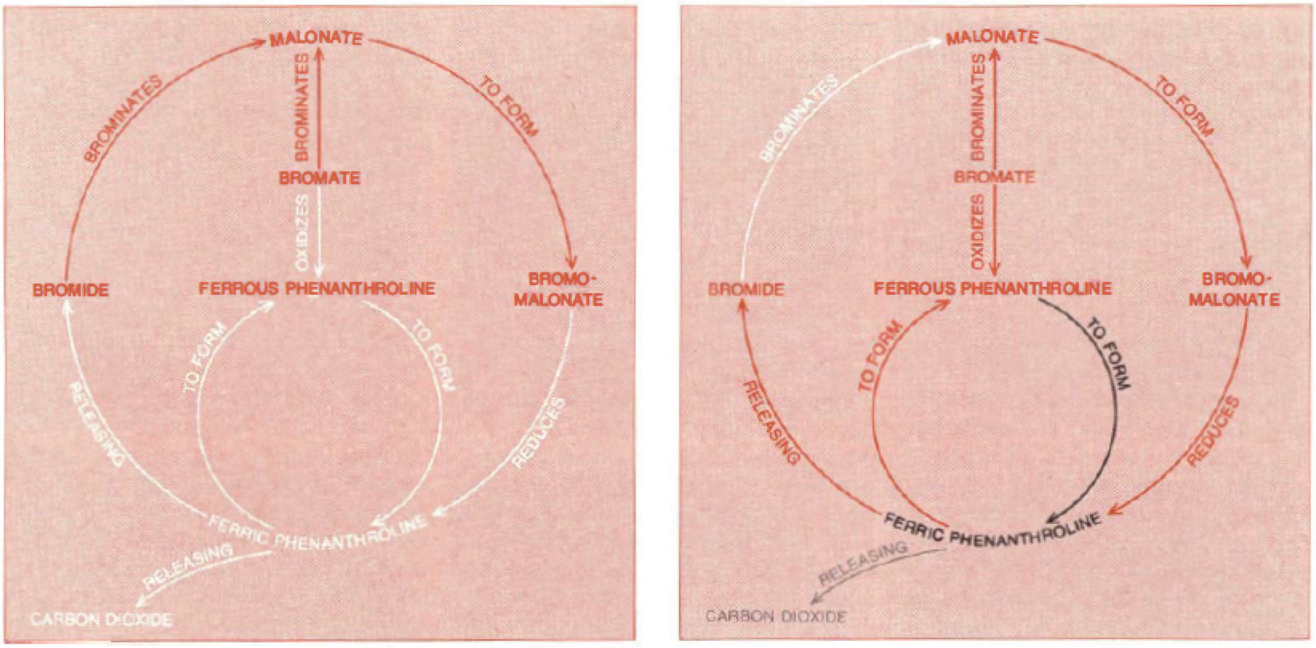

Figure 6.6: Outline of th BZ reaction [Fig. of Winfree Sci Am 1974 p82]

Fig. 6.6 TWO SETS OF REACTIONS can account for the oscillation of ferroin from red to blue and back to red. The concentration of Br^- determines which of the two sets of reactions will dominate. In the first set (left) the bromide and bromate both brominate malonate to form bromomalonate. During this process ferroin (II) is red. If the concentration of the bromide drops below a threshold level, then the second set of reactions (right) starts to dominate. The last vestige of bromide is con sumed and the $BrO_3{}^-$ takes over the bromination of the malonate. Simultaneously it oxidizes ferroin changing it from red to blue. Accumulated bromomalonate now reduces ferroin(III) back to its red form ferrous, releasing Br^- and carbon dioxide. High concentration of bromide shuts off

in any established journals. Even after this reaction became famous, he never showed up in any meeting on the reaction. I am sympathetic to Belousov who quit science for the reason that science (or the science community) was not scientific enough. S. E. Shnoll was interested in the reaction, to whom Belousov gave the prescription and promised to publish his report in the annual report of the institute he belonged to. Shnoll told his student Zhabotinski to study the mechanism.

[91] A = $BrO_3{}^-$, B = all oxidized organic species, X = $HBrO_2$, Y = Br^-, Z = feroin(III).

this reaction sequence and restarts the red stage.

A + Y $\longrightarrow$ X + P
X + Y $\longrightarrow$2P
A+ X $\longrightarrow$2X + 2Z
2X $\longrightarrow$A + P
B + Z $\longrightarrow$$(f/2)$Y

A further simplification is possible, because Y is slaved to other concentrations. Eventually, we get

$$\varepsilon\dot{x} = x(1-x) + f(q-x)z/(q+x) = g(x,z), \;\; \dot{z} = x - z = h(x,z). \tag{6.10}$$

Here $\varepsilon \simeq 10^{-2}$ and $q \sim 10^{-3}$. Let us simplify these further to

$$\varepsilon\dot{x} = x(1-x) - fz = g(x,z), \;\; \dot{z} = x - z = h(x,z). \tag{6.11}$$

However, this is an oversimplification, because $x < 0$ must not happen. For very small x ($x < q$) the sign of the coefficient of z must be negative. Thus, a simplified model must be

$$\varepsilon\dot{x} = x(1-x) - f(x)z = g(x,z), \;\; \dot{z} = x - z = h(x,z). \tag{6.12}$$

where $f(x) = f - a\delta(x)$ ($a < 0$) for $x \geq 0$ to prevent x falling to the negative world.

The fixed points are $x = z$, $x^2 - (1-f)x = 0$. Therefore, $(x,z) = (0,0)$ and $(1-f, 1-f)$ are fixed points. The former is a saddle:

$$\frac{d}{dt}\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} 1/\varepsilon & -f/\varepsilon \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix}. \tag{6.13}$$

Around the other fixed point, we have

$$\frac{d}{dt}\begin{pmatrix} \delta x \\ \delta z \end{pmatrix} = \begin{pmatrix} (2f-1)/\varepsilon & -f/\varepsilon \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \delta x \\ \delta z \end{pmatrix}. \tag{6.14}$$

Notice that the characteristic equation is $\lambda^2 - \lambda \mathrm{Tr}\, A + \mathrm{det} A = 0$. In our case $\mathrm{det} A = (1-f)/\varepsilon$ and $(1/\varepsilon)\mathrm{Tr}\, A = 2f - 1 - \varepsilon$. Assume $f < 1$. Then, the eigenvalues are complex. Therefore, (ignoring the small ε) for $f = 1/2$ the fixed point is a center. If $f \in (1/2, 1)$ $\mathrm{Tr}\, A > 0$, so orbits near the fixed point spiral out. This bifurcation is called a Hopf bifurcation as we will discuss later.

6.16 Nullcline approach

Do we have a limit cycle? In this case the flow is certainly confined: x and z mut be positive, and cannot be too large. There is no attracting fixed point anywhere. To see the situation closer, a good way is to draw the nullclines $g = 0$ and $h = 0$ (Fig. 6.7).

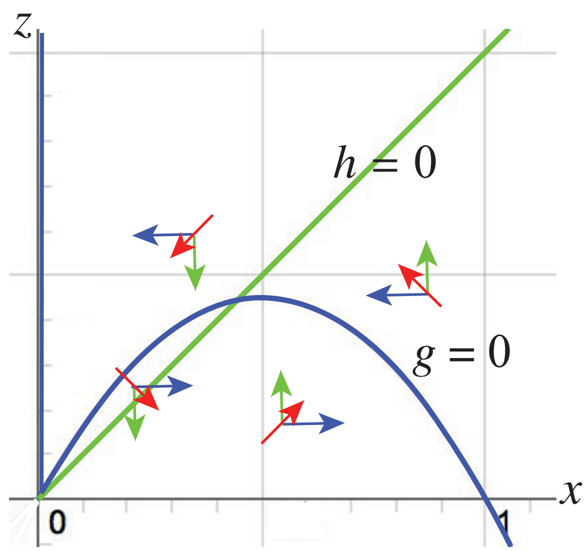

Figure 6.7: Nullclines and vector field for a simplified Oregonator.