5 Lecture 5: Hyperbolicity

5.1 Hyperbolic fixed points

We have already introduced a singular point x of a vector field $X \in \mathcal{X}^r(M)$, where $r \geq 1$ and M is (usually) a compact n-manifold: $X(x) = 0$. Its derivative at x is a linear operator $D_x X = A$. A is a $n \times n$ matrix[71].

If A has eigenvalues whose real parts are nonzero (i..e, not neutral), we say x is a hyperbolic fixed point.

5.2 Invariant set

For a flow ϕ_t for $\mathcal{X}^r(M)$ (or map $f \in C^r(M)$), a subset $S \subset M$ is an invariant set, if for any $x \in S$, $\phi_t(x) \in S$ for all $t \in \mathbb{R}$ ($f^n(x) \in S$ for any $n \in \mathbb{Z}$).

Needless to say, singular points (= fixed points) are in the invariant set of the dynamical system.

5.3 Non-wandering set[72] A nonwandering point of a dynamical system $\mathcal{X}^r(M)$ (or map $f \in C^r(M)$) is a point such that for its any nbh U there is t such that $\phi_t(U) \cap U \neq \emptyset$ ($n \in \mathbb{Z}$ such that $f^n(U) \cap U \neq \emptyset$). The totality of non-wandering points is the non-wandering set of the dynamical system.

5.4 ω and α limit sets

Long time behaviors of a dynamical system is studied by limit sets describing the $t \to \pm\infty$ behaviors of trajectories.

ω-limit set of x: the set of accumulation points of $\phi_t(x)$ for $t > 0$. That is,
$\omega(x) = \{y \mid \lim_i \phi_{t_i}(x) = y, t_i \to \infty\}$.
α-limit set of x: the set of accumulation points of $\phi_t(x)$ for $t < 0$. That is,
$\alpha(x) = \{y \mid \lim_i \phi_{t_i}(x) = y, t_i \to -\infty\}$.

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[71] Of course, we choose an appropriate chart, but as we have discussed at length in Section 2, we may assume our world is a (bounded) subset of $\mathbb{R}^n$.

[72] There are several different definitions. For example,
(i) x is a non-wandering point if there is a nbh U and τ such that $\phi_t(U) \cap U \neq \emptyset$ for $t > \tau$.
(ii) x is a non-wandering point if for any nbh U there is $t > 0$ such that $\phi_t(U) \cap U \neq \emptyset$.
Notice that $\phi_{-t}(\phi_t(U) \cap U) = U \cap \phi_{-t}(U)$, so the direction of time does not matter.

5.5 Some properties of limit sets

The following statement should be intuitively clear.

Proposition. ω and α limit sets of a point x are non-wandering closed sets. [Wandering sets are open.]

Proposition. If N is a invariant set, then ∂N, N°, $\overline{N}$ and N^c are invariant. [This is due to the continuity of ϕ_t.]

Proposition. The totality of the non-wandering sets is an invariant set.

Any point in periodic orbits and fixed points is non-wandering but there are more subtle nonwandering points. If an orbit densely fill a domain, then any point in(side) the domain is non-wandering, although it need not be periodic nor fixed point.

Proposition. For any point in the non-empty compact invariant set, its α and ω limit sets are non-empty. [If x_0 is in a compact invariant set, then $\{\phi_t(x_0)\}$ is in a compact set, so its accumulation points are in the same compact set.]

5.6 Attracting set

A closed invariant set $A \subset M$ is an attracting set, if there is a nbh of A such that any $x \in U$ stays in U (i.e., $\phi_t(x) \in U$ for all $t > 0$) and $\phi_t(x) \to A$.

The domain of attraction of A (basin of A) is $\cup_{T \leq 0} \phi_t(U)$.

By reversing time $t \to -t$ we can analogously define repelling sets.

Even in 1D an attracting set can be complicated as the following example by Ruelle shows:

$$\dot{x} = -x^4 \sin \frac{\pi}{x}. \tag{5.1}$$

5.7 Hyperbolic linear vector field

If the spectrum of $L \in \mathcal{L}(\mathbb{R}^n)$ is disjoint from the imaginary axis, L is called hyperbolic. The number of eigenvalues with negative real part is called the index of L. Hyperbolic linear fields are open dense in $\mathcal{L}(\mathbb{R}^n)$.

If L is hyperbolic, then there are invariant subspaces E^s and E^u such that $\mathbb{R}^n = E^s \oplus E^u$. $L|_{E^u}$ has positive real eigenvalues and $L|_{E^s}$ has negative real eigenvalues.

Two hyperbolic linear vector fields are topologically conjugate iff their indices are identical.

5.8 Hyperbolicity

For a hyperbolic fixed point p we can linearize the dynamics, and then study the linearized system as a dynamics on $\mathbb{R}^n$. Its eigenspace for stable eigenvalues (on the left half space for vectors and inside the unit circle for maps) may be understood as a tangent subspace E^s of T_pM. We can collect trajectories tangent to the vectors that is a locally invariant submanifold of M. This is the local stable mfd for p denoted as $W_U^{s\,\mathrm{loc}}(p)$, where U is an appropriate nbh of p (see **5.9**).

Reversing the time we can define a local unstable manifold as well.

We have seen in **4.4** that $\mathcal{G}_0$ the vector fields with only simple singularities) is open-dense in $\mathcal{X}^r(M)$.[73] We can show that

Theorem. The vector fields $\mathcal{G}_\tau$ whose singularities are all hyperbolic is open-dense in $\mathcal{X}^r(M)$.[74]
[Demo']
We have only to show that $\mathcal{G}_\tau$ is open-dense in $\mathcal{G}_0$, but at least intuitively this should not be surprising.

5.9 Local stable manifold

Let p be a fixed point of $f \in C^r(M, M)$ and U be a neighborhood. The local stable manifold $W_U^{\mathrm{loc}}(p)$ in U is given by

$$W_U^{s\,\mathrm{loc}}(p) = \{q \in U \mid f^n(q) \in U \text{ for } \forall n \in \mathbb{N}\}. \tag{5.2}$$

5.10 Stable manifold

The stable manifold $W^s(p)$ of p is defined as

$$W^s(p) = \cup_{n=0}^{\infty} f^{-n}\left(W_U^{s\,\mathrm{loc}}(p)\right) \tag{5.3}$$

That is, the totality of the points eventually mapped to p is its stable manifold.
The stable manifold of f^{-1} is the unstable manifold.
Remark: $W^s(p)$ need not a a submanifold of M.

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[73] PdM p56
[74] PdM p58

5.11 Stable manifold theorem

Let p be a hyperbolic fixed point of $f \in C^r(M, M)$. Thus, $T_pM = V^s \oplus V^u$, where V^s (resp. V^u) is the vector space on which T_pf is contracting (resp., expanding). Then, there is a contraction $g : W^s(p) \to W^s(p)$ and embedding $J : W^s(p) \to M$ such that $Jg = fJ$. Furthermore, $T_pJ : T_p(W^s(p)) \to V^s$ is an isomorphism.

5.12 Example: Renormalization group flow in the Hamiltonian space

We may interpret the renormalization group transformation as a map from a (generalized) canonical distribution μ to another (generalized) canonical distribution $\mu' = \mathcal{R}\mu$. We can imagine effective Hamiltonians H and H' (it is customary that β is absorbed in H's) according to

$$\mu = \frac{1}{Z}e^{-H}, \ \mu' = \frac{1}{Z'}e^{-H'}. \tag{5.4}$$

We may write $H' = \mathcal{R}H$. Therefore, we can imagine that successive applications of $\mathcal{R}$ defines a flow (RG flow) in the space of Hamiltonians (or models or systems). This idea is illustrated in Fig. 5.1.

In Fig. 5.1 H^* is a fixed point with an infinite correlation length of the RG flow. Its stable manifold is called the *critical surface*. The Hamiltonian of the actual material, say, magnet A, changes (do not forget that β is included in the definition of the Hamiltonian in (5.4)) as the temperature changes along the trajectory denoted by the curve with 'magnet A.' It crosses the critical surface at its critical temperature. The renormalization transformation uses the actual microscopic Hamiltonian of magnet A at various temperatures as its initial conditions. Three representative RG flows for magnet A are depicted. 'a' is slightly above the critical temperature, 'b' exactly at T_c of magnet A ('b′' is the corresponding RG trajectory for magnet B, a different material; both b and b′' are on the critical surface), 'c' slightly below the critical temperature. Do not confuse the trajectory (black curve) of the actual microscopic system as temperature changes and the trajectories (successive arrows; RG flow) produced by the RG transformation.

If we understand H^*, we understand all the universal features of the critical behaviors of all the magnets crossing its critical surface.

`http://www.youtube.com/watch?v=MxRddFrEnPc` a video by Douglas Ashton.

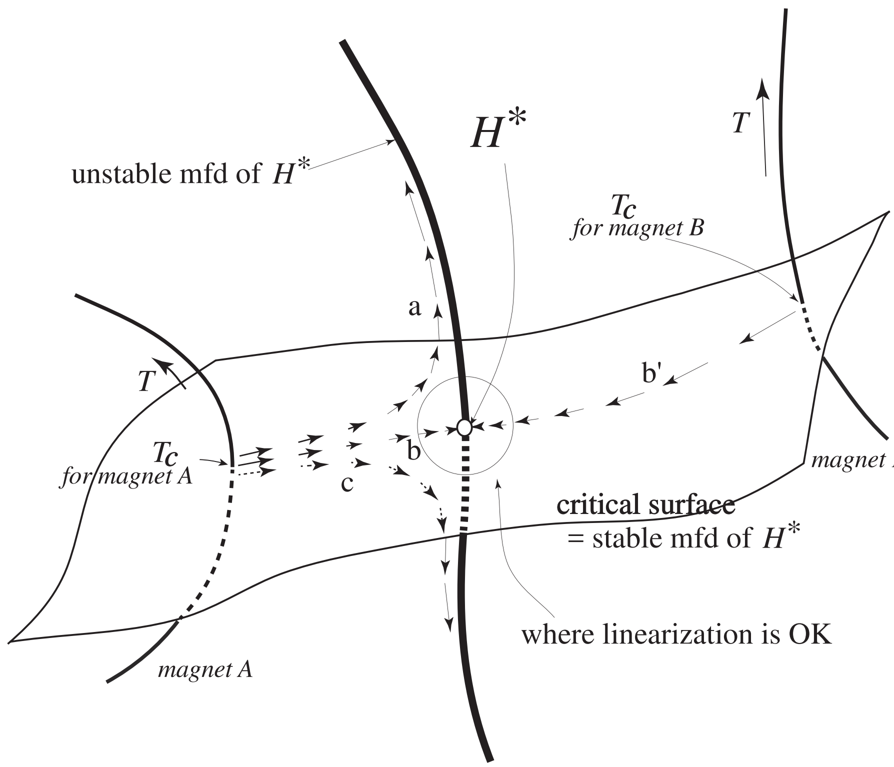

Figure 5.1: A global picture of renormalization group flow in the Hamiltonian space $\mathcal{H}$. The explanation is in the text. 'mfd' = manifold. The thick curves emanating from H^* denote the direction that the Hamiltonians are driven away from the fixed point by renormalization.

5.13 Hartman's theorem
Let $X \in \mathcal{X}^r(M)$ and $p \in M$ be a hyperbolic singularity of X. Then, X is locally equivalent to its linearization.

Here equivalence means topological conjugacy. That is, for $A = DX_p$, there is a continuous map h such that $hx(t) = e^{At}hx_0$.

Notice that h is a homeo, not a diffeo. If we wish to have a diffeomorphic conjugation, then there is a strong relation between the two flow velocities, but such relations are already fixed by the two vector fields we are comparing. Thus, such h may not be chosen.[75]

To prove this theorem is to construct h.

5.14 Strategy to show Hartman's theorem
The linearized system has $L_t = e^{At}$ as the evolution operator. The evolution operator for the original system may be written as a sum of L_t and the deviation from it ϕ_t (i.e., $\varphi_t = L_t + \phi_t$).
(i) There is a homeo h such that $h(L_\tau + \phi_\tau) = L_\tau h$ for some (perhaps small) τ.
(ii) The following H

$$H = \int_0^\tau e^{-As} h\varphi_s ds \tag{5.5}$$

is actually $H = h$ and $H\varphi_s = L_s H$ for $\forall s \in \mathbb{R}$.

Let us show the last statement holds, assuming we have constructed h. Its construction is in **5.15**-.

$$L_{-s}H\varphi_s = L_{-s}\int_0^\tau L_{-t}h\varphi_t dt \varphi_s = \int_0^\tau L_{-t-s}h\varphi_{t+s}dt. \tag{5.6}$$

Let us introduce $u = t + s - \tau$. Then,

$$\begin{aligned}
\int_0^\tau L_{-t-s}h\varphi_{t+s}dt &= \int_{s-\tau}^s du\, L_{-u-\tau}h\varphi_{u+\tau} &(5.7)\\
&= \int_{s-\tau}^0 du\, L_{-u-\tau}h\varphi_{u+\tau} + \int_0^s du\, L_{-u-\tau}h\varphi_{u+\tau} &(5.8)\\
&= \int_{s-\tau}^0 du\, L_{-u-\tau}h\varphi_{u+\tau} + \int_0^s du\, L_{-u}L_{-\tau}h\varphi_\tau\varphi_u &(5.9)
\end{aligned}$$

[75] See a comment on p33-4 of Palis-de Melo.

$$= \int_{s-\tau}^{0} du\, L_{-u-\tau} h \varphi_{u+\tau} + \int_{0}^{s} du\, L_{-u} h \varphi_{u}. \tag{5.10}$$

We have used $h(L_\tau + \phi_\tau) = h\varphi_\tau = L_\tau h$. Introduce $v = u + \tau$. Thus, we have shown

$$L_{-s} H \varphi_s = \int_{s}^{\tau} dv\, L_{-v} h \varphi_v + \int_{0}^{s} du\, L_{-v} h \varphi_v = H. \tag{5.11}$$

This equality is true for $s = \tau$, so $H = h$.

5.15 Formal construction of h

Let $h = 1 + u$

$$(1 + u)(L_\tau + \varphi_\tau) = L_\tau(1 + u) \tag{5.12}$$

implies

$$\mathcal{L}u \equiv L_\tau u - u(L_\tau + \phi_\tau) = \phi_\tau. \tag{5.13}$$

We write

$$\mathcal{L} = L_\tau \mathcal{L}^* \text{ with } \mathcal{L}^* u = u - L_\tau^{-1} u(L_\tau + \phi_\tau). \tag{5.14}$$

Since L_τ is invertible and $L_\tau + \varphi_\tau$ is a homeomorphism, $\mathcal{L}^*$ is invertible. Thus, u exists. We have shown that the formal solution is actually real. However, we have not guaranteed that h is homeomorphism. that is, invertible.

5.16 h is invertible

We must show that h is a homeo: can we invert $1 + u$? u must be small: we must estimate

$$\|u\| \le \|L_\tau^{-1}\| \|(\mathcal{L}^*)^{-1}\| \|\phi_\tau\|. \tag{5.15}$$

$\|L_\tau^{-1}\|$ is bounded. Therefore, we must show $\|(\mathcal{L}^*)^{-1}\|$ is bounded and $\|\phi_\tau\|$ must be small.

5.17 $\|(\mathcal{L}^*)^{-1}\|$ is bounded

Let us write $\mathcal{L}^* - 1 = \mathcal{K}$. Actually $\mathcal{K}u = -L_\tau^{-1} u(L_\tau + \phi_\tau)$. $\mathcal{K}$ is invertible: Notice that formally

$$\mathcal{K}^{-1} u = L_\tau u(L_\tau + \phi_\tau)^{-1}. \tag{5.16}$$

Since $L_\tau + \phi_\tau$ is the time evolution operator for the system, it is at least locally homeomorphic. Therefore, its inverse is well defined. Notice that $\mathcal{K}$ is a linear map keeping the stable (E^s) and unstable (E^u) subspaces at 0 intact.[76] $\mathcal{K}$.

Thanks to the hyperbolicity $\|\mathcal{K}\| \le a < 1$ on the stable subspace thanks to the 'shrinking nature' of L_τ. Also $\|\mathcal{K}^{-1}\| \le a < 1$ on the unstable subspace thanks to the 'expanding nature' of L_τ. Then,
(a) $1+\mathcal{K}$ is isomorphic on the stable subspace and $\|(1+\mathcal{K})^{-1}\| \le 1/(1-a)$.
(b) $1+\mathcal{K}$ is isomorphic on the unstable subspace and $\|(1+\mathcal{K})^{-1}\| \le a/(1-a)$.
Thus, $1+\mathcal{K} = \mathcal{L}^*$ is isomorphic and $\|(\mathcal{L}^*)^{-1}\| \le \max\{1/(1-a), a/(1-a)\} = 1/(1-a)$.[77]In that case We must show (a) and (b).

(a): To compute the norm consider $(1+\mathcal{K})^-1y = x$ for y in the tangential space of the stable manifold with $\|y\| = 1$. Then $y = x + \mathcal{K}x$ implies $1 \ge \|x\| - \|\mathcal{K}\|\|x\|$ or $\|x\| \le 1/(1-\|\mathcal{K}\|)$. Thus, $\|(1+\mathcal{K})^{-1}\| \le 1/(1-a)$.
(b): Analogously, $(1+\mathcal{K})^-1y = \mathcal{K}^{-1}(1+\mathcal{K}^{-1})^-1y = x$ implies $y = (1+\mathcal{K}^{-1})\mathcal{K}x$. Therefore, $1 \ge (1-\|\mathcal{K}^{-1}\|)\|x\|/\|\mathcal{K}^{-1}\| = (1-a)\|x\|/a$. Thus, $\|(1+\mathcal{K})^{-1}\| \le a/(1-a)$.

5.18 $\|\phi_\tau\|$ is small for small τ

We make $\varphi_\tau = L_\tau + \varphi_\tau$. We assume X is Lipschitz with some constant K. (3.25) in **3.23** tells us

$$\|\varphi_t(x) - \varphi_t(y)\| \le e^{Kt}\|x-y\|. \tag{5.17}$$

We must evaluate $\phi_t = \varphi_t - L_t$. Solving the equations formally, we get

$$\varphi_t(x) = x + \int_0^t A\varphi_s(x)ds + \int_0^t \psi(\varphi_s(x))ds, \tag{5.18}$$

$$L_t(x) = x + \int_0^t AL_s(x)ds, \tag{5.19}$$

where we write $X = Ax + \psi$. Therefore,

$$\phi_t(x) = \int_0^t A[\varphi_s(x) - L_s(x)]ds + \int_0^t \psi(\varphi_s(x))ds = \int_0^t A\phi_s(x)ds + \int_0^t \psi(\varphi_s(x))ds. \tag{5.20}$$

Hence, we obtain

$$\|\phi_t(x) - \phi_t(y)\| \le \int_0^t \|A(\phi_s(x) - \phi_s(y))\|ds + \int_0^t \|\psi(\varphi_s(x)) - \psi(\varphi_s(y))\|ds. \tag{5.21}$$

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[76] $L_\tau + \phi_\tau$ is a homeomorphism at least locally, so it keeps the stable and unstable manifolds intact. Thus, their tangent spaces at the fixed point are intact. These tangent spaces are just eigensubspaces of L_τ.

[77] Here, a convenient norm satisfying $\|x+y\| \le \max\{\|x\|, \|y\|\}$ is chosen for $x \in E^s$ and $y \in E^u$ (as in most books), but the usual one is OK if you allow a constant multiple in front of $1/(1-a)$.

Now we use (5.17) and the Lipschitz property of ψ with sufficiently small constant δ to get

$$\|\psi(\varphi_s(x)) - \psi(\varphi_s(y))\| \le \delta\|\varphi_s(x) - \varphi_s(y)\| \le \delta e^{Kt}\|x - y\| \tag{5.22}$$

Therefore, (5.21) has the Gronwall form:

$$\|\phi_t(x) - \phi_t(y)\| \le \int_0^t \delta e^{Ks}\|x - y\|ds + \|A\| \int_0^t \|\phi_s(x) - \phi_s(y)\|ds. \tag{5.23}$$

Or if δ (and τ) is small enough, we can choose a small positive number ε and have

$$\|\phi_\tau(x) - \phi_\tau(y)\| \le \varepsilon\|x - y\| + \|A\| \int_0^\tau \|\phi_s(x) - \phi_s(y)\|ds. \tag{5.24}$$

We use Gronwall's inequality **3.22**

$$\|\phi_\tau(x) - \phi_\tau(y)\| \le \varepsilon e^{\|A\|\tau}\|x - y\| \tag{5.25}$$

or

$$\|\phi_\tau(x)\| \le \varepsilon e^{\tau\|A\|}\|x\|. \tag{5.26}$$

Thus, we have shown that ϕ_τ is Lipschitz and sufficiently small.

5.19 What if singularity is not hyperbolic? Center manifold[78]

If the fixed point (at 0) we consider is not hyperbolic, then the linearization gives us a matrix with vanishing real parts. Let us consider still the stable case. That is, the fixed point is not stable linearly but thanks to higher order terms the point is a ω-limit point of itself.

In such a case the ODE looks like

$$\begin{aligned} \dot{x} &= Ax + f(x, y), &(5.27)\\ \dot{y} &= By + g(x, y), &(5.28) \end{aligned}$$

where x and y correspond to the neutral and stable subspaces: All the eigenvalues of A have no real part, and the eigenvalues of B have negative real parts. f and g are higher order terms and vanish at the origin. If $f = g = 0$ then, $x = 0$ is a stable

[78] A good introduction is J. Carr *Applications of canter manifold theory* (Springer1981).

manifold, and $y = 0$ is called a center manifold. More generally, if $y = h(x)$ is an invariant manifold, it is called a center manifold (which is not unique, generally). At least locally, a center manifold exists which is C^2. Let us proceed formally.

We solve

$$\dot{y} = h'(x)\dot{x} \ \Rightarrow, \ Bh(x) + g(x,h) = h'(x)(Ax + f(x,h)). \tag{5.29}$$

to determine h. $h(0) = h'(0) = 0$ is the auxiliary condition.

On $y = h(x)$ the flow u is governed by

$$\dot{u} = Au + f(u, h(u)). \tag{5.30}$$

Thus, the dynamics is reduced to the one on a lower dimensional space.

How can we obtain h? Solving (5.29) is equivalent to solving the original system. However, if we can solve (5.29) approximately, we can get a reasonable approximation to h. Set

$$M(h) = Bh(x) + g(x,h) - h'(x)(Ax + f(x,h). \tag{5.31}$$

If $M(\phi) = O[|x|^q]$ $(q > 1)$, then $|h - \phi| = O[|x|^q]$.

5.20 Lyapunov stability

A time-independent solution (stationary solution, fixed point p) of an autonomous differential equation is said to be Lyapunov stable, if all the trajectories starting from a neighborhood of p is defined for all $t > 0$ and converges uniformly in time to p.

Thus, $t \to \infty$ behavior need not be a convergence to p. More formally, for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$\|p - x(0)\| < \delta \ \Rightarrow \ \|p - x(t)\| < \varepsilon \text{ for } \forall t > 0, \tag{5.32}$$

we say the fixed point p is Lyapunov stable.

5.21 Asymptotic stability

A fixed point p is asymptotically stable, if it is Lyapunov stable and $\lim_{t\to\infty} x(t) = p$ for some nbh of p.

Look at counterexamples in Fig. 5.2.

As can be seen from these examples, convergence to an equilibrium point in the $t \to \infty$ limit of all the solutions starting at near p is not a sufficient condition for its

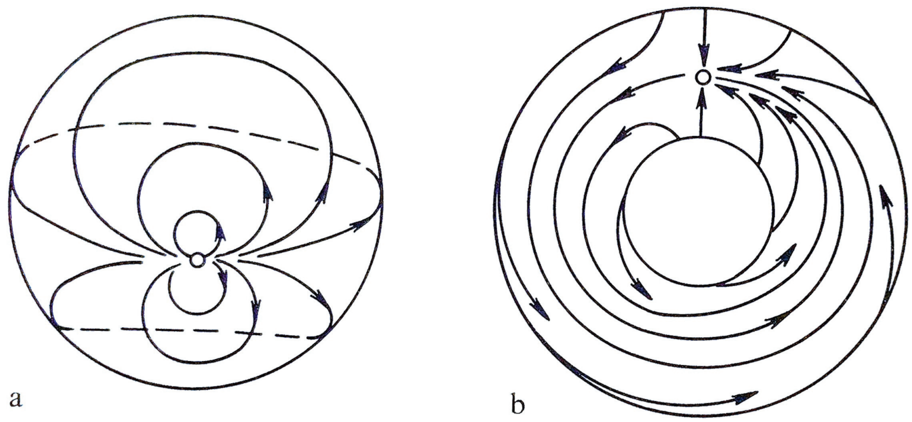

Figure 5.2: Unstable singular points to which all the trajectories which start at nearby points converge. [Fig. 3 of Arnold DS I]

asymptotic stability.

Note that a center is Lyapunov stable, but not asymptotically stable. In such a case the stability is called marginal.

5.22 Stability by linearization

If all the eigenvalues of the linearized equation at a singular point have negative real parts then the singular point is asymptotically stable.

This is a special case of what Hartman's theorem implies.

5.23 Lyapunov function[79]

A differentiable function f is called a Lyapunov function for a singular point x_0 of a vector field X if $Xf \le 0$[80] and x_0 is its strict local minimum in a neighborhood of x_0.

5.24 Lyapunov's stability theorem A singlarity of a differentiable vector field for which a Lyapunov function exists is stable.

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[79] A DS I p24; For other attractors we can define an analogous concept. See p202-3 of DS1 by Anosov.

[80] Note that $(d/dt)f(x) = Xf(x)$; We use the 'standard notation' $X = \sum X_i \frac{\partial}{\partial x_i}$.