4 Lecture 4: Singularity

4.1 General study of flow of vector field

We have studied what happens in the domain where there is no singularity of X. We can rectify local patches and then connect them maximally to obtain a unique flow. Now, we are left with singularities.

Since X vanishes there, the flow should be slow in the neighborhood, so we have only to study it in some small neighborhood of the singularities. Thus, often linearization of the original X around singularities is effective.

Especially when the singularity is hyperbolic (i.e., linearized vector field may be expressed in terms of linear operator whose spectrum is not on the imaginary axis), the linearized system and the original system are homeomorphic (Hartman's theorem), so we may study the stability of singularities by linearization.

4.2 Simple singularity[53]

$p \in M$ is a simple singularity of $X \in \mathcal{X}^r(M)$ if $DX_p : T_pM \to T_pM$ does not have zero as an eigenvalue.[54]; in other words, the linearized vector field at p may be written as Ax ($A = DX_p$) and A is non-singular.

4.3 Simple singularities are isolated

M as in **4.2**. All the C^r-vector fields on M sufficiently close to X has a simple singularity near p. The position of the zero depends continuously on the vector field.[55] This should be intuitively clear, since the solution to $X = 0$ is locally unique, because A is non-singular, and the solution should depend continuously on X.

Since the derivative of X 'does not vanish', X and the vector field everywhere 0 (= the zero section of TM) should cross transversally.[56] Thus, the isolation of simple singularities should be obvious.

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[53] Palis-de Melo p55-

[54] hyperbolicity $\Rightarrow$ simplicity, but not vice versa.

[55] An official expression is: There exist a neighborhood $\mathcal{N}(X)$, a neighborhood U_p of p and a continuous function $\rho : \mathcal{N}(M) \to U_p$ such that all $Y \in \mathcal{N}(M)$ has a unique zero $\rho(Y) \in U_p$. Palis-de Melo Proposition 3.1.

[56] Proposition 3.2 of Palis-de Melo.

4.4 Vector fields with only simple singularities are structurally stable

More precisely, vector fields $\mathcal{G}_0$ with only simple singularities are open dense in $\mathcal{X}^r(M)$. This follows from **4.3**. Hyperbolic vector fields are also open dense.

The concept for diffeo f corresponding to simplicity is being elementary: if 1 (i.e., identity) is not an eigenvalue of D_pf, f is elementary.

4.5 Linearization of ODE around singularity

For $\dot{(x)} = X(x)$ with a smooth X, near its singularity X should be small, so if we take the singularity at the origin, x is small and $\dot{x}$ is small. Thus, it is a natural idea that the linearized equation

$$\dot{x} = Ax \tag{4.1}$$

with

$$A = \frac{dX}{dx}_{x=0} \tag{4.2}$$

can tell us the local behavior of the system near the singularity.

The justification of the idea is not always possible, but if A has no pure imaginary eigenvalues (the so-called hyperbolic case),[57] the idea goes through. Therefore, let us study linear systems (4.1) fairly in detail first.

You must sense the logical error in the following argument physicists always use: "Let us assume $\|x\|$ is small near the singularity p. Then, we may use linearization (4.1) around p. Since all the eigenvalues have negative real parts, we may conclude p is a stable fixed point," but we have assumed from the start that δx doe not grow (and small to linearize the system!).

4.6 Exponential function of linear operators

If a linear operator is bounded (that is, $\sup_{\|x\|=1}\|Ax\| = \|A\| < \infty$)

$$e^A = \sum_{n=0}^{\infty}\frac{A^n}{n!} \tag{4.3}$$

is well defined ($\|e^A\| \le e^{\|A\|}$).

if $[A, B] = 0$, then

$$e^A e^B = e^{A+B} = e^B e^A. \tag{4.4}$$

[57]Such matrices are open dense. Cf,˙ **5.7**.

Using this, the general solution to (4.1) reads

$$x(t) = e^{At}x_0. \tag{4.5}$$

To compute the matrix representation of e^A it is convenient to 'diagonalize' A. However, unless A is normal ($AA^* = A^*A$) (and also allow the use of complex eigenvalues), this is not possible. If A is defined on a complex vector field, we can still use the Jordan normal form (see appendix). Therefore, we use a trick called 'complexification.'

4.7 Complexification[58]

Since eigenvalues of real matrices are generally complex, if we wish to use linear operator theory fully, it is convenient to consider A as an operator on $\mathbb{C}^n$ instead of its original domain $\mathbb{R}^n$. To realize this, we introduce 'complexification' $\mathcal{C}$.

Since any vector in $\mathbb{C}^n$ may be written as $a + ib$ ($a, b \in \mathbb{R}^n$), we can complexify $A \to \mathcal{C}(A) = \tilde{A}$ according to

$$\mathcal{C}(A)(a + ib) = Aa + iAb. \tag{4.6}$$

Since $\mathcal{C}$ is linear by definition, to preserve the algebraic (i.e., the ring) structure of matrices, we should show the following:

$$\mathcal{C}(AB) = \mathcal{C}(A)\mathcal{C}(B). \tag{4.7}$$

From this we know

$$\mathcal{C}(e^A) = e^{\mathcal{C}(A)}. \tag{4.8}$$

We can also show (recall the definition of the norm)

$$\|\mathcal{C}(A)\| = \|A\|. \tag{4.9}$$

4.8 Complex and real diagonalization of real matrix

We know normal matrices (satisfying $AA^* = A^*A$) may be unitary diagonalized on

[58] A very kind explanation is found in M. W. Hirsch and S. Smale, *Differential equations, dynamical systems and linear algebra* (Academic Press, 1974), Chapter 4.

the complex vector space.[59] However, this is not generally possible on the real vector space. The eigenvectors corresponding to λ and $\overline{\lambda}$ e_1 and e_2 (respectively):

$$\tilde{A}e_1 = \lambda e_1, \quad \tilde{A}e_2 = \overline{\lambda}e_2 \tag{4.10}$$

cannot be chosen in $\mathbb{R}^n$. Notice that we may choose $e_2 = \overline{e_1}$, so we choose two real vectors $a = e_1 + e_2$ and $b = (e_1 - e_2)/i$ among the basis.[60] Diagonalization of A is not possible, but still 'almost diagonalized' real matrix may be obtained.

To make the situation crisp clear, consider a 2×2 matrix that can be diagonalized on $\mathbb{C}^2$ as

$$\begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix} \tag{4.11}$$

with the basis $\{e_1, e_2\}$. $\lambda = \alpha + i\beta$. If we use the basis $\{a, b\}$ the above diagonalized matrix reads

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \tag{4.12}$$

Since the 'real translation' is straightforward, we consider everything on $\mathbb{C}^n$.

4.9 Complexification of linear ODE

(4.1) on $\mathbb{R}^n$ can be complexified as

$$\dot{z} = \mathcal{C}(A)z, \tag{4.13}$$

where $z = x + iy$. This consists of two real equations $\dot{x} = Ax$ and $\dot{y} = Ay$.

How to recover the real solution from the full complexified computation is explained in **4.8**.[61]

4.10 Singularity of 2-real vector field

The singularity of 2-dimensional system may be classified according to the Jordan normal form of A.

(1) 0 is a sink: stable fixed point.

$$\text{(a)} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} (\lambda, \mu < 0), \quad \text{(b)} \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, (a < 0, b \neq 0), \quad \text{(c)} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} (\lambda < 0). \tag{4.14}$$

[59] Normality is a necessary and sufficient condition for A to be diagonalized by a unitary transformation.

[60] If you wish to normalize them, divide them with $\sqrt{2}$.

[61] Detailed examples can be seen in Hirsch+Smale, so I will not dwell on examples.

For case (b) the origin is called a focus. (c) is a bicritical node.[62]

(2) 0 is a source: in any direction it is unstable.

$$\text{(d)} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} (\lambda, \mu > 0), \quad \text{(e)} \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, (a > 0, b \neq 0), \quad \text{(f)} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} (\lambda > 0). \tag{4.15}$$

For case (e) the origin is called a focus. (f) is a bicritical node.

(3) 0 is a saddle: there is one stable direction and one unstable direction.

$$\text{(g)} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} (\lambda > 0 > \mu). \tag{4.16}$$

These are hyperbolic cases. The phase portraits look like Fig. 4.1.

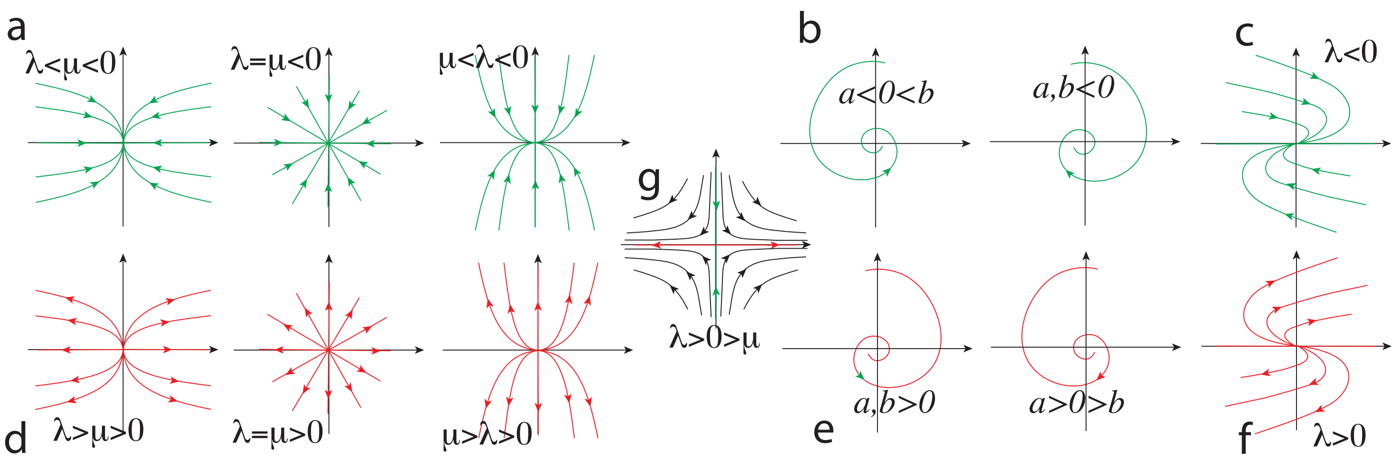

Figure 4.1: Linear flows on the plane

The non-hyperbolic cases cannot be classified without referring to the higher order terms. That is, linearization does not preserve qualitative nature of the singularity.

(4) Non-hyperbolic case.

$$\text{(h)} \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} (b \neq 0), \quad \text{(i)} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{4.17}$$

[62]In this case the flow looks, e.g., like $x = e^{t\lambda}x_0 + te^{t\lambda}y_0$, $y = e^{t\lambda}y_0$.

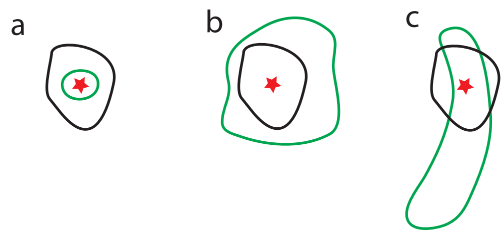

Figure 4.2: Another illustration of (hyperbolic) singularities Black 'before', Green, 'after' The start is the critical point = fixed points. a: sink, b: source, c: saddle

Classification (see Fig. 4.3 for an example): https://media.pearsoncmg.com/aw/ide/idefiles/media/JavaTools/lnclppan.html. This demo inevitably includes non-simple zero cases. Watch what happens.

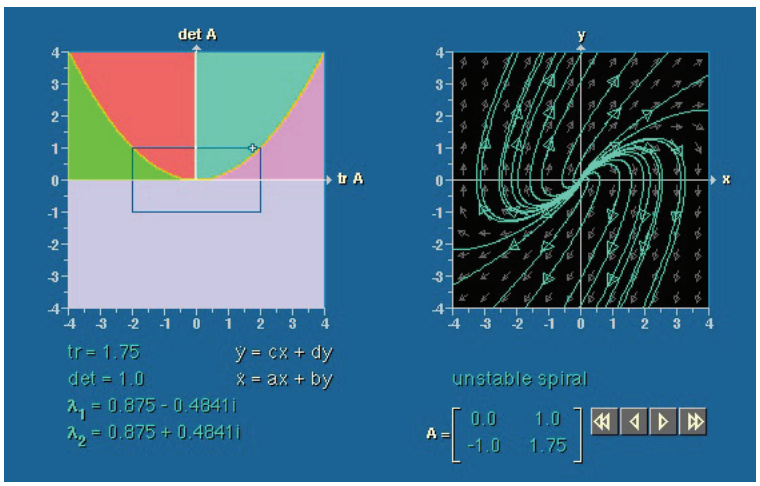

Figure 4.3: Classification

Appendix 1. Jordan normal form

4.11 Generalized eigenspace[63]

The eigenvalues are the roots of the characteristic equation

$$\det(A - xI) = \prod_k (\lambda_k - x)^{n_k} = 0. \tag{4.18}$$

[63] A very kind explanation of the Jordan normal form is found in M. W. Hirsch and S. Smale, *Differential equations, dynamical systems and linear algebra* (Academic Press, 1974), Chapter 6.

Here n_k is the multiplicity of λ_k. $\mathrm{Ker}(A-\lambda_k I) = V_k$ is called the eigenspace of λ_k.[64]

$\mathrm{Ker}(A-\lambda_k I)^{n_k} = V_k$ is called the generalized eigenspace of A belonging to λ_k.

4.12 The fundamental decomposition theorem

Let A be a linear operator on $V = \mathbb{C}^n$. Then, the dimension of the generalized eigenspace V_k is n_k and $\oplus_k V_k = V$.

This is proved in **4.13** and **4.14**.

4.13 Basic decomposition lemma

The key part of **4.12** is: for any linear $A : V \to V$

$$V = M \oplus N, \tag{4.19}$$

where $M = \cap_{j\in\mathbb{N}^+} A^j V$ and $N = \cup_{j\in\mathbb{N}^+}\mathrm{Ker}(A^j)$.[65]
[Demo] Let $M_j = A^j V$ and $N_j = \mathrm{Ker}(A^j)$. Then,

$$\begin{aligned} 0 &= N_0 \subset N_1 \subset \cdots \subset N_j \subset N_{j+1} \subset \cdots \subset V, && (4.20)\\ V &= M_0 \supset M_1 \supset \cdots \supset M_j \supset M_{j+1} \supset \cdots \supset 0. && (4.21) \end{aligned}$$

Since V is finite dimensional, there must be $n, m \in \mathbb{N}^+$ such that $j \geq m \Rightarrow M_j = M_m$ and $j \geq n \Rightarrow N_j = N_n$. Let $M = M_m$ and $N = N_n$. Since $A^n M = M$, if $x \in M$ is not zero, then $A^n x \neq 0$. On the other hand, $A^n N = 0$, so for any $y \in N$ $A^n y = 0$. Therefore, $M \cap N = \{0\}$.

For any $y \in V$ let $x \in M$ such that $A^m x = A^m y$; such x exists, because $A^m y \in M$ and A^m is reversible on M. Then, $z = y - x$ is in $N_m \subset N$, so $V = M \oplus N$.

4.14 Concluding demo of fundamental decomposition theorem

Instead of A, let us consider $A_k = A - \lambda_k I$. Let us define $M_k = \cap_{j\in\mathbb{N}^+} A_k^j V$ and $N_k = \cup_{j\in\mathbb{N}^+}\mathrm{Ker}(A_k^j)$. Applying the fundamental decomposition theorem to A_1, we get $V = N_1 \oplus M_1$.

Now, restricting A_2 on M_1 we can repeat the argument above. $M_1 = N_2 \oplus M_2'$, where M_2' is the orthogonal complement of N_2 in M_1. Thus, we would arrive at

$$V = \oplus_k N_k. \tag{4.22}$$

If we prove $N_k = V_k$, we are done. Obviously $V_k \subset N_k$. This follows from the following Lemma. We need one definition:

N is a nilpotent operator if for some $m \in \mathbb{N}^+$ $N^m = 0$ (in the domain). Notice that $A_k = A - \lambda_k I$ is a nilpotent operator, if restricted to N_k.

Lemma. Let n be the smallest positive integer such that $N^n x = 0$ for $x \in V$. Then, $\{x, Nx, \cdots, N^{n-1}x\}$ is a basis of subspace $Z = \{x, Nx, \ldots, N^j x, \cdots\}$ (called a cyclic subspace) of V.

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[64] If $\oplus_k V_k = \mathbb{C}^n$, we say A is unitary diagonalizable. Notice that, generally speaking, even if you collect all the eigenvectors, the result can only span a genuine subset of the original vector space.

[65] Here, the demonstration follows Hirsch-Smale, but if you know the basic theorem of linear algebra $\mathrm{Im}(A) + \mathrm{Ker}(A) = V$ for any linear operator A, it is obvious.

[Demo] Since $N^n x = 0$, obviously $Q = \{x, Nx, \cdots, N^{n-1}x\}$ can generate Z. Thus, we have only to show that Q is linearly independent. If not,

$$\sum_{k=1}^{n-1} a_k N^k x = 0 \tag{4.23}$$

has a nontrivial solution with the first nonzero coefficient a_j. Operating N^{n-j-1} we have

$$a_j N^{n-1} x + N^{n-j-1} \sum_{k=j+1}^{n-1} a_k N^k x = a_j N^{n-1} x = 0. \tag{4.24}$$

This contradicts the definition of n.

4.15 $\boldsymbol{A}$ on $\boldsymbol{V}_k$

Since V is decomposed into the direct sum of V_k, we have handle each subspace separately. Let us consider A restricted on V_k. Let $S = \lambda_k I$ (here I is n_k-dimensional identity) and $N = A - \lambda_k I$. N is nilpotent, $A = N + S$ and $SN = NS$. This decomposition is unique.

This implies that V can be decomposed into the diagonal S and nilpotent N uniquely as $A = N + S$, $NS = SN$ on V.

The lemma in **4.14** tells us that we can choose $U = \{x, Nx, \cdots N^{n_k-1}x\}$ as the basis of V_k. The subspace spanned by $U_{-1} = \{x, Nx, \cdots N^{n_k-2}x\}$ is one dimension smaller than V_k, so we can choose a vector y in V_k but orthogonal to this smaller space U_{-1}. Now $Ny \in U_{-1}$ and is orthogonal to y, Notice that Ny is not in $U_{-2} = \{x, Nx, \cdots N^{n_k-3}x\}$ but may not be orthogonal to U_{-2}. Thus we make y' such that $Ny = Ny' + z$, where Ny' is orthogonal to U_{-2} and $z \in U_{-2}$. Repeating this argument, we can make $N^j q$ orthogonal to $N^k q$ ($j \neq k$). That is, we can choose q such that $\{q, Nq, \cdots, N^{n_k-1}q\}$ makes an orthogonal basis. With this basis, N is expressed as an $n_k \times n_k$ matrix:

$$N = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ & & & \cdots & & & \\ & & & \cdots & & & \\ & & & \cdots & & & \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \tag{4.25}$$

This means A on V_k has the following $n_k \times n_k$ matrix:

$$A = \begin{pmatrix} \lambda_k & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & \lambda_k & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \lambda_k & \cdots & 0 & 0 & 0 \\ & & & \cdots & & & \\ & & & \cdots & & & \\ & & & \cdots & & & \\ 0 & 0 & 0 & \cdots & 1 & \lambda_k & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & \lambda_k \end{pmatrix} \tag{4.26}$$

This is called a Jordan cell.

4.16 Jordan normal form
A we have been considering can be represented as the direct sum of the Jordan cells. The number of cells with the same eigenvalue λ is given by dim Ker$(A - \lambda I)$.

4.17 Detailed example[66]

Let us solve the following linear ODE $\dot{x} = Tx$ that reads wrt a coordinate system as :

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} -1 & 1 & -2 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{pmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{4.27}$$

Let us write

$$T_0 = \begin{pmatrix} -1 & 1 & -2 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.28}$$

The solution is $e^{T_0 t}\boldsymbol{x}_0$, where $\boldsymbol{x}_0$ is the initial condition. We wish to calculate this explicitly. Notice that

$$\begin{pmatrix} -1 & 1 & -2 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ -2 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 6 & -9 & -2 \\ -9 & 17 & 4 \\ -2 & 4 & 1 \end{pmatrix} \tag{4.29}$$

which is not equal to

$$\begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ -2 & 4 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 & -2 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 2 & -6 \\ 2 & -8 & 21 \end{pmatrix} \tag{4.30}$$

so we cannot diagonalize T_0 with an orthogonal transformation. The eigenvalues are obviously -1, -1 and 1. The rank of

$$T_0 + 1 = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 4 \\ 0 & 0 & 2 \end{pmatrix} \tag{4.31}$$

[66] Hirsh-Smale Ex2 Chapter 6, but all the details are filled so that you do not need any pencil nor paper for follow all the details.

is 2, so its kernel is 1 dimension spanned by $(1,0,0)^T$.[67] Thus the eigenvalue -1 has algebraic multiplicity 2 and geometrical multiplicity 1. Thus we need the generalized eigenspace of -1 defined by

$$0 = (T_0+1)^2\boldsymbol{x} = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 4 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 4 \\ 0 & 0 & 2 \end{pmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 8 \\ 0 & 0 & 4 \end{pmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{4.32}$$

Therefore, the generalized eigenspace is spanned by $(1,0,0)^T$ and $(0,1,0)^T$. The (generalized) eigenspace of 1 is given by

$$0 = (T_0-1)\boldsymbol{x} = \begin{pmatrix} -2 & 1 & -2 \\ 0 & -2 & 4 \\ 0 & 0 & 0 \end{pmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{4.33}$$

This implies that z is arbitrary, say 1, and $-y+2z=0$ implies $y=2$ and $x=0$: Thus, $\{|a\rangle, |b\rangle, |c\rangle\} = \{(1,0,0)^T, (0,1,0)^T, (0,2,1)^T\}$ is a basis (not orthogonal!) wrt which $T = S + N$, where S is diagonal $-1 \oplus -1 \oplus 1$ and N is nilpotent according to the general theorem.

Let us rewrite T wrt to this basis (which is denoted as T_1). $PT_1 = T_0P$ or $T_1 = P^{-1}T_0P$. We need[68]

$$\langle a|T|b\rangle = \sum_{x,y}\langle a|x\rangle\langle x|T|y\rangle\langle y|b\rangle \tag{4.34}$$

Notice that

$$P = \langle y|b\rangle = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.35}$$

Therefore, P^{-1} is given by (Note that $\det P = 1$)

$$P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.36}$$

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[67] Here we have used the basic theorem: for a linear map T from a vector space V to another vector space W, dim V = dim(Im T) + dim(Ker T).

[68] We use the bra-ket notation for the 'mnemonic sake'; since our kets are not orthonormal, the 'transposition' is actually to compute the transposition of the inverse matrix. The mnemonics works perfectly. Look at

$$\sum_x \langle a|x\rangle\langle x|b\rangle = \delta_{ab}.$$

$(P)_{xb} = \langle x|b\rangle$, $(P^{-1})_{ax} = \langle a|x\rangle$.

Thus, $T_1 = P^{-1} T_0 P$:

$$T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} T_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & -2 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.37}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.38}$$

Thus

$$S_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.39}$$

Obviously, $[S_1, N_1] = 0$. This implies

$$N_0 = P N_1 P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.40}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.41}$$

This tells us that

$$T_0 = \begin{pmatrix} -1 & 1 & -2 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{pmatrix} = S_0 + N_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.42}$$

Notice that, as the general theory tells us, $[S_0, N_0] = 0$. Therefore,

$$e^{tT_0} = e^{tS_0} e^{tN_0}. \tag{4.43}$$

Since N_0 is nilpotent (actually $N_0^2 = 0$:

$$\begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.44}$$

we have

$$e^{tN_0} = 1 + tN_0 = \begin{pmatrix} 1 & t & -2t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.45}$$

To compute e^{tS_0}, we should use

$$e^{tS_0} = Pe^{tS_1}P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{t} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.46}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} & -2e^{-t} \\ 0 & 0 & e^{t} \end{pmatrix} = \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} & -2e^{-t}+2e^{t} \\ 0 & 0 & e^{t} \end{pmatrix} \tag{4.47}$$

Thus we have arrived at the final answer:

$$e^{tT_0} = \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} & -2e^{-t}+2e^{t} \\ 0 & 0 & e^{t} \end{pmatrix} \begin{pmatrix} 1 & t & -2t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{-t} & te^{-t} & -2te^{-t} \\ 0 & e^{-t} & -2e^{-t}+2e^{t} \\ 0 & 0 & e^{t} \end{pmatrix} \tag{4.48}$$

Appendix 2. Degree theory

The aim of this appendix is very restricted: to understand the degree of the vector field with only simple singularities. Illustrations are only in 2-space.

4.18 Index of singularity
Take a ball B containing only one singularity of a vector field $X : B \to \mathbb{R}^n$. Thus, X has no singularity on $\partial B = S^{d-1}$ if the vector field is d-dimensional. $\deg(X, B)$ is defined as the degree of the map $f : \partial B \to \mathbb{R}^d$, which is defined as follows:

$$\deg(X, B) = p - q, \tag{4.49}$$

where p (resp., q) is the number of points in $f^{-1}(x)$ ($x \in \partial B$) with det df being positive (resp., negative).

4.19 Examples of simple singularities
For 2-vector fields,
The indices of a sink, a source or a center is $+1$.
The index of a hyperbolic saddle is -1.
See Fig. 4.4.

The map $f : D \to \mathbb{R}^2$, where D is a disk around the origin of the base space, corresponds to the relation between the red and the black arrows. Thus, to obtain the degrees for 2-space examples, we have only to see how many times the black arrows rotate when the base vector rotate once (i.e., when we ground ∂D once).

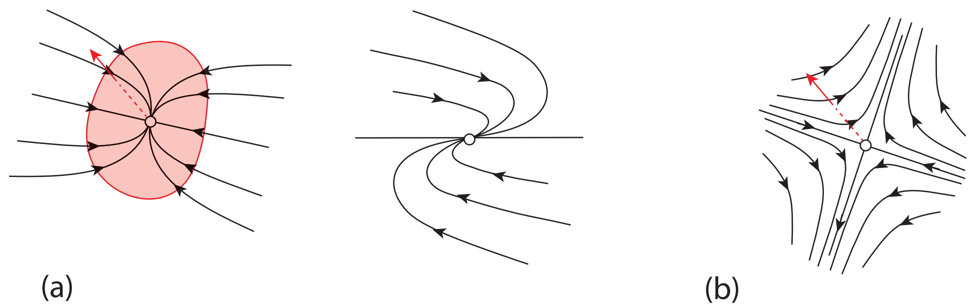

Figure 4.4: (a) with index +1, (b) −1. Red arrows show the base space vectors.

4.20 Total sum of indices = degree

The degree of a vector field X with simple singularities is defined as

$$\deg(X, D) = \sum_{s\in D} \text{index} s, \tag{4.50}$$

where s is a singularity.

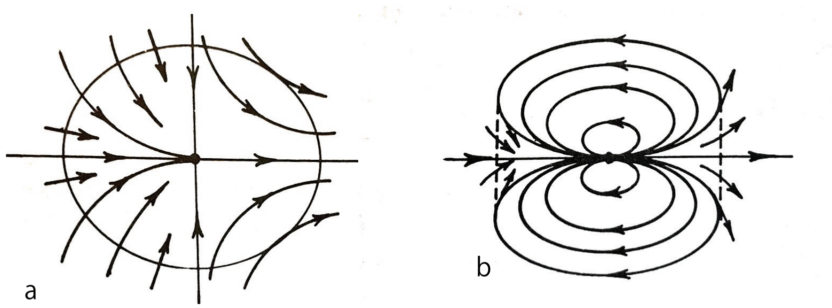

Figure 4.5: (a) with degree 0 (index zero), (b) index +2.

In Fig. 4.5 (a) we may interpret the zero due to merging of a sink and a saddle. Thus, $+1 + (-1) = 0$ is the degree of the disk in the figure. We get the same answer even if we follow the original definition. For (b) we could follow the rotation of the vector along a small circle around the singular point (or from the merging of a sink and a source).

4.21 Poincaré-Hopf's theorem on degrees

Let M be a compact orientable manifold, and X be a differentiable vector field on M with finitely many singular points. Then,

$$\sum_{z\in M} \text{index}(z) = \chi(M), \tag{4.51}$$

where the sum is over all the singular points on M, and $\chi(M)$ is the Euler index[69] That is, the degree of $\mathcal{X}^r(M)$ is identical to the Euler index of M.
[Demo]
If we accept that the LHS of (4.51) does not depend on X on M, then we can compute it with a convenient X. For a polyhedron assign a sink to a vertex, a source to a surface, and saddle to an edge (Fig. 4.6).

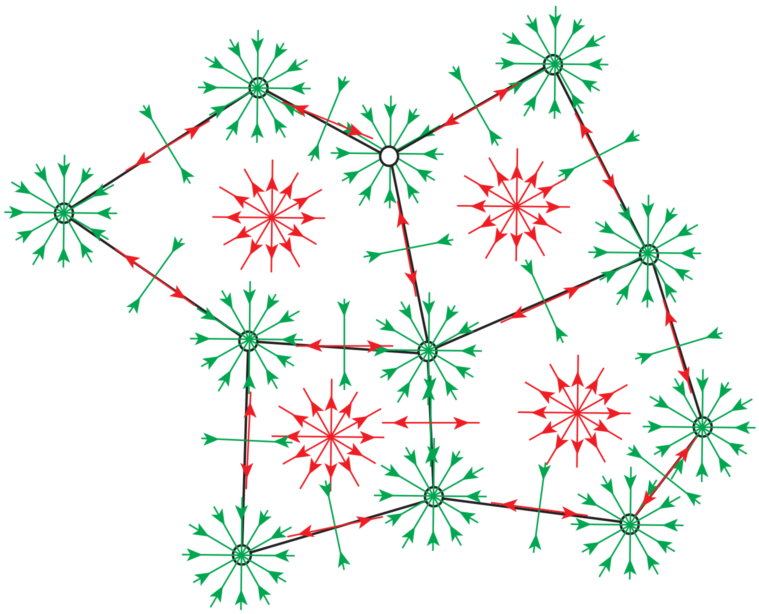

Figure 4.6: Demo of Poincare-Hopf theorem

We can construct X. For this obviously (4.51) holds.

[69]or Euler characteristics defined by

$$\chi(M) = \sum_{i=0}^{\infty}(-1)^i b_i,$$

where b_i is the ith Betti number. For a CW complex,

$$\chi(M) = \sum_{i=0}^{\infty}(-1)^i n_i,$$

where n_i is the number of i-simplexes in M. In particular, we get $\chi(M) = V - E + F + \cdots + (-1)^n X_n + \cdots = 1$, where V: number of vertices, E: number of edges, F: number of faces, B: number of 3-simplexes. In 3-space of a polyhedron (surface) this is always 2. More generally, $\chi - 2 - 2g$, where g is the genus of the manifold (the number of handles attached to the sphere or the number of holes; Solid torus has $g = 1$.

4.22 Degree-theoretical constraints on singularities

Poincare-Hopf's theorem imposes a strong constraints on the existence of various singularities of the vector field on a manifold. For example, if the vector field is on S^2, it is impossible to have a single source or saddle. If there is a saddle, there must be at least three other sources/sinks. However, on T^2 a single source-saddle pair can live happily.

In Fig. 4.5 (a) can be on T^2 without any other singularity, and (b) on S^2. See Fig. 5.1.