44 Palis conjecture

44.1 Stability conditions

As we have seen, Axiom A alone is not enough to guarantee the structural stability. Thus, Smale conjectured that:

Axiom A + no-cycle condition implies $C^1$-$\Omega$-stability.\(^{457}\)

This was proved by Smale (for diffeos) and by Pugh & Shub (for flows). Another conjecture is

Axiom A + strong transversality implies $C^1$-stability.

This was proved by Robbin and Robinson.

As in the case of Peixoto’s theorem to show the necessity is much harder: For a system to be stable the above mentioned conditions are needed. Mañé proved the necessity for diffeomorphisms.\(^{458}\) Hayashi showed that the $C^1$-dynamical system case: a $C^1$-dynamical system must be hyperbolic to be stable.\(^{459}\)

A high point in Hayashi’s work is his connecting lemma\(^{460}\) creating homoclinic orbits by $C^1$-small perturbations of flow or diffeomorphism an unstable manifold accumulating on some stable one can be $C^1$-perturbed to make it intersect one another (the creation of homoclinic or heteroclinic orbits).

We usually claim that a phenomenon of a system relevant to natural science must be structurally stable, because science demands reproducibility. If we stick to this ‘dictum,’ we have only to study ‘nice’ Axiom A systems. However, we already know if the system dimension is not too low (1 for diffeo and 2 for flow as Peixoto showed and horseshoes cap.), we have something else as an open set. The members of such an open set is structurally unstable: we have ‘stable structural instability.’ There must be corresponding natural phenomena.

44.2 Palis’ global conjecture on metric stability of attractors

I: There is a $C^r$-dense set $D$ of dynamical systems such that any element of $D$ has finitely many attractors whose union of basins of attraction has total probability.

II: The attractors of the elements in $D$ support a physical measure.

\(^{456}\)Hayashi 1997; J Palis, A GLOBAL VIEW OF DYNAMICS AND A CONJECTURE ON THE DENSENESS OF FINITUDE OF ATTRACTORS

\(^{457}\)That is, the nonwandering sets are preserved.


\(^{460}\)This is explained in his review
III: These properties are metrically stable (i.e., in $D$ any $k$-parameter small $C^\nu$-perturbation preserves the dynamics).

IV: The attractors are stochastically stable. (Against noise)

V: For 1D maps the attractors are either sinks or abs cont measures.

### 44.3 Palis conjecture 1993

1. Every $C^\nu$-diffeo of a compact mfd $M$ can be $C^\nu$-approximated by one of the following:
   (a) a hyperbolic system (Axiom A with strong transversality)
   (b) a system with heterodimensional cycle
   (c) a system exhibiting a homoclinic tangency.

2. If $M$ is 2D then (a) or (c) occurs (in other words, Palis conjectured that avoiding homoclinic bifurcation, the generalized Peixoto’s picture can be recovered).
   For $r = 1$ Pujals and Sabarino proved 2.\textsuperscript{463}, 1 is still open.
   for $r \geq 2$ both are wide open. For $r \geq 2$ this is widely open. even for 2D.

\textsuperscript{461} Lyubitch showed this for quadratic maps.

\textsuperscript{462} A. Katok, B. Hasselblatt, Handbook of dynamical systems vol 1A(2005); E. R. Pujals From Peixoto’s theorem to Palis’s conjecture (2009).