44 Palis conjecture

44.1 Stability conditions[456]

As we have seen Axiom A alone is not enough to guarantee the structural stability. Thus, Smale conjectured that:

Axiom A + no-cycle condition implies C^1-Ω-stability.[457]

This was proved by Smale (for diffeos) and by Pugh & Shub (for flows). Another conjecture is

Axiom A + strong transversality implies C^1-stability.

This was proved by Robbin and Robinson.

As in the case of Peixoto's theorem to show the necessity is much harder: For a system to be stable the above mentioned conditions are needed. Mañé proved the necessity for diffeomorphisms.[458] Hayashi showed that the C^1-dynamical system case: a C^1-dynamical system must be hyperbolic to be stable[459] A high point in Hayashi's work is his connecting lemma[460] creating homoclinic orbits by C^1-small perturbations of flow or diffeomorphism an unstable manifold accumulating on some stable one can be C^1-perturbed to make it intersect one another (the creation of homoclinic or heteroclinic orbits).

We usually claim that a phenomenon of a system relevant to natural science must be structurally stable, because science demands reproducibility. If we stick to this 'dictum,' we have only to study 'nice' Axiom A systems. However, we already know if the system dimension is not too low (1 for diffeo and 2 for flow as Peixoto showed and horseshoes cap.), we have something else as an open set. The members of such an open set is structurally unstable: we have 'stable structural instability.' There must be corresponding natural phenomena.

44.2 Palis' global conjecture on metric stability of attractors

I: There is a C^r-dense set D of dynamical systems such that any element of D has finitely many attractors whose union of basins of attraction has total probability.

II: The attractors of the elements in D support a physical measure.

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[456] Hayashi 1997; J Palis, A GLOBAL VIEW OF DYNAMICS AND A CONJECTURE ON THE DENSENESS OF FINITUDE OF ATTRACTORS

[457] That is, the nonwandering sets are preserved.

[458] R Mañé, A proof of the C^1-stability conjecture, Publ Math IHES 66 161 (1988).

[459] S Hayashi Connecting invariant manifolds and the solution of the C^1 stability and Ω-stability conjectures for flows, Annals of Math 145 81 (1997).

[460] This is explained in his review

III: These properties are metrically stable (i.e., in D any k-parameter small C^r-perturbation preserves the dynamics).
IV: The attractors are stochastically stable. (Against noise)
V: For 1D maps the attractors are either sinks or abs cont measures.[461]

44.3 Palis conjecture 1993[462]

1. Every C^r-diffeo of a compact mfd M can be C^r-approximated by one of the following:
   (a) a hyperbolic system (Axiom A with strong transversality)
   (b) a system with heterodimensional cycle
   (c) a system exhibiting a homoclinic tangency.
2. If M is 2D then (a) or (c) occurs (in other words, Palis conjectured that avoiding homoclinic bifurcation, the generalized Peixotos picture can be recovered).

For $r = 1$ Pujas and Sabarino proved 2.[463], 1 is still open.

for $r \geq 2$ both are wide open. For $r \geq 2$ this is widely open. even for 2D.

[461] Lyubitch showed this for quadratic maps.

[462] A. Katok, B. Hasselblatt, Handbook of dynamical systems vol 1A(2005); E. R. Pujals From Peixoto's theorem to Palis's conjecture (2009).

[463] E. Pujals and M. Sambarino, Homoclinic tangencies and hyperbolicity for surface diffeomorphisms, Ann. of Math. 151 (2000), 961-1023.