43 Heterodimensional cycles

43.1 Preliminary definitions[450]

The limit set $L(f)$ of f is

$$L(f) = \overline{\cup_{x \in M}(\alpha(x) \cap \omega(x))} \tag{43.1}$$

A point x is nonwandering if for every neighbourhood U of x there exists m, $m \neq 0$, such that $f^m(U) \cap U \neq \emptyset$. These points form the nonwandering set $\Omega(f)$. Obviously, $L(f) \subset \Omega(f)$.[451]

43.2 Axiom A related definitions and terminology

A diffeomorphism f satisfies Axiom A if $\Omega(f) = \overline{\mathrm{Per}(f)}$, and $\Omega(f)$ is hyperbolic. In this case $L(f) = \Omega(f)$.

There is a spectral decomposition: $\Omega(f) = \cup \Omega_i$, where Ω_i is f-invariant, transitive (= with a dense orbit), local maximal (i.e., there is a nbh U_i of Ω_i such that $\Omega_i = \cap_{\mathbb{Z}} f^n(U_i)$) and compact. Moreover $\Omega_i = \overline{H(P)}$, where $H(p)$ is the transversal homoclinic points related to P (i.e, $H(P) = W^s(P) \overline{\pitchfork} W^u(P)$), where $P \in \Omega_i \cap \mathrm{Per}(f)$. Ω_i is called a basic set, and U_i isolating nbh of Ω_i.

The index of Ω_i is defined by $\dim W^s(P)$, where P is any periodic point in Ω_i. The index does not depend on the choice of P in Ω_i.

43.3 Local stability due to hyperbolicity

Hyperbolicity implies local stability: given a basic set Ω and its isolating neighbourhood U for any g C^r-close to f, $r > 1$, $\Omega(g)$ is hyperbolic and there is an homeomorphism $h : \Omega \to \Omega(g)$. $\Omega_g)$ is called continuation of Ω.

’43.4 Ω-stability

f is said to be Ω-stable if it has a conjugate continuation in its sufficiently small C^r nbh.

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[450] Lan Wen *Differentiable Dynamical Systems* An Introduction to Structural Stability and Hyperbolicity.

[451] Recall Bowen's non SBR counterexample.

43.5 We cannot ignore behaviors off Ω[452]

This is because we must worry about the mutual relations among basic sets. A simple examples are[453]

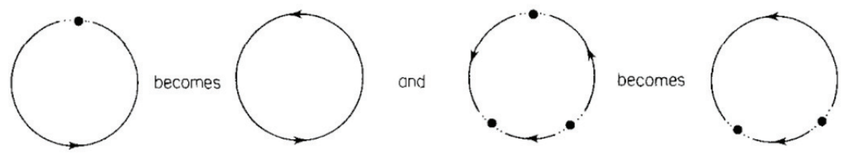

Figure 43.1: Simple Ω-explosion [Irwin Fig. 7.32]

Inn Fig. 43.1 Left, there is a one-way zero (non-hyperbolic zero). It is a the Ω set of the system. With a small perturbation we can remove it. The outcome is that the whole S^1 is Ω now, and example of the Ω-explosion. 43.1 Right, three fixed points reduce to two ('implosion').

43.6 Global explosion of Axiom A systems

Look at Smale's example (a diffeo in S^2): Its Ω consists of six hyperbolic fixed points Fig. 43.2.

We must see that the saddle connections are not in Ω, they are wandering. For example, take p. Its nbh eventually goes to sinks c or d.

Now, look at colored arrows in the figure. If we make a small surgery to cross $W^u(x)$ and $W^s(y)$, then p becomes non-wandering. Thus, $W^u(x) \cap W^s(y)$ is now non-wandering. However, $W^s(x) \cap W^u(y)$ is still wandering. Now, the Ω after perturbation consists of the previous fixed points + the new saddle connection.

43.7 Cycles Let M be a closed C^∞-manifold and consider $\mathcal{X}^r(M)$ $(r \geq 1)$. We say that $X, Y \in \mathcal{X}^r(M)$ are Ω-conjugate if there is a homeomorphism $h : \Omega(X) \to \Omega(Y)$ sending trajectories of X into those of Y. $X \in \mathcal{X}(M)$ is Ω-stable if for any $\varepsilon > 0$ there is a neighborhood $N(X)$ in $\mathcal{X}^r(M)$ such that if $Y \in N(X)$ then X is Ω-conjugate to

[452] Proc. Symp. Pure Math. Vol. 14, Amer. Math. Soc: Rhode Island, 1970.

[453] M C Irwin, *Smooth Dynamical Systems* (World Scientific, 2001) p185

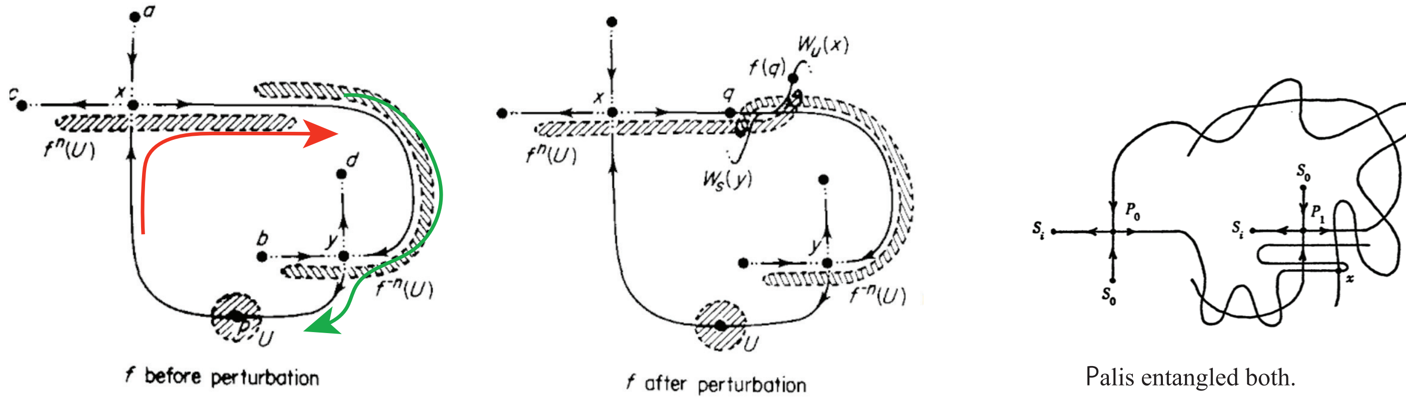

Figure 43.2: A is a global source and c a global sink. [7.35] Rightmost from Palis PAMS paper

Y by a homeomorphism which is ε-C^0 close to the identity map in $\Omega(X)$.

For an Axiom A system for each basic set we can define its stable and unstable manifolds.

There is an n-cycle on Ω, if there is a sequence of basic sets $\Omega_0, \cdots, \Omega_{n-1}$ with $W_0 = W_n$, $\Omega_i \neq \Omega_j$if $i \neq j$ and

$$W^s(\Omega_i) \cap W^u(\Omega_{i+1}) = \emptyset. \tag{43.2}$$

A cycle is called equidimensional if index Ω_i making the cycle are identical and heterodimensional otherwise.

43.8 Ω-explosion[454]

For Smale's Axiom A' system:
(i) Ω is the disjoint union of the set of critical points F and the closure Λ of its periodic orbits,
(ii) each element of F is hyperbolic and Λ is a hyperbolic set.
Theorem: If X satisfies Axiom A' and there is a cycle on Ω, then X is not Ω-stable.

[454] J. Palis, Ω-explosion, Proc AMS 27 85 (1971). The diffeo version is J. Palis, A note on Ω-stability, Proc. Sympos. Pure Math., vol. 14, Amer. Math. Soc, Providence, R. I., 1970. In this paper Palis gives a sufficient condition for Ω-stability as well for special cases: If Ω is the finite union of hyperbolic critical points and closed orbits and has the no-cycle property, then X is Ω-stable.

That is, no-cycle condition is a necessary condition for Ω-stability.

Notice that so far we discussed the existence of explosive or dangerous cases

43.9 Stable non-Axiom A cycles[455]

Let M be a 3-mfd.

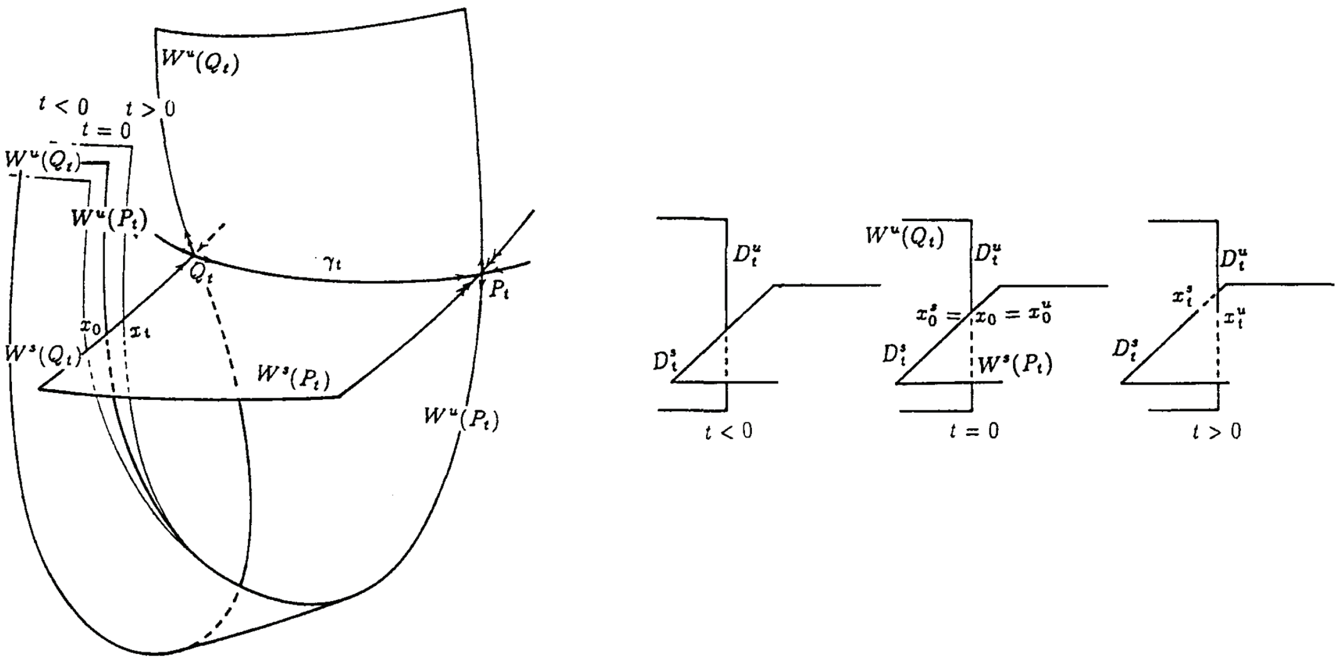

Figure 43.3: [Diaz Fig.1]

(I) Connected interesection
(1) $\dim W^s(P_0)) = 2$, $\dim W^u(Q_0) = 1$.
(2) There is an $_0$ invariant curve $\gamma_0 \subset W^s(P_0)) = 2 \pitchfork W^u(Q_0)$.
From now on $0 \to t$ indicates continuations.
(II) Creation and generic unfolding of the cycle
There are C^1 curves: $x_t \in W^s(Q_t)$ and $x_t \in W^s(Q_t)$. (III) Strong foliation condition.

[455] L J Diaz, Ribust nonhyperbolic dynamics and heterodimensional cycles, Ergod. Th. & Dynam. Sys. 15 291 (1995).

THEOREM 1. Let f_t satisfy the above conditions. Then for $t \in [0, t_0]$
(1) $\gamma_t \subset L(f_t)$, so $L(f_t)$ is not hyperbolic,
(2) f_t is Ω-stable. That is, f_t can be C^∞-approximated by a diffeo exhibiting a heterodimensional cycle.

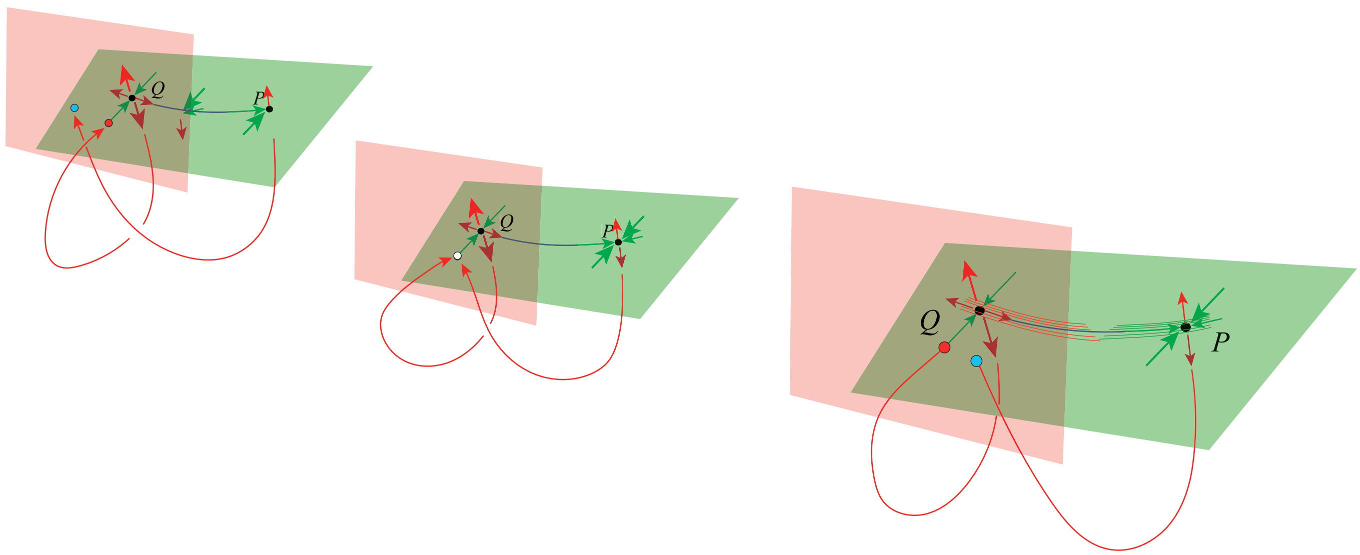

Figure 43.4: [Diaz Fig.1]