42 Newhouse phenomenon

42.1 Homoclinic point in modified linear map

The following an ex[licit example of a homoclinic point found in Palis-de Melo textbook.

Consider $\varphi(x,y) = (2x, y/2)$ and $\Psi(x,y) = (x - f(x+y), y + f(x+y))$, where f is continuous $f(x) = 0$ for $x < 1$ and $f(2) > 2$. Then $\Psi \circ \varphi$ has a homoclinic point on the y-axis between $y = 1$ and 2.

42.2 Homoclinic bifurcation

If a horseshoe map is slid as in Fig. 42.1, a non-transversal homoclinic orbit is formed. (b) has a homoclinic tangency.

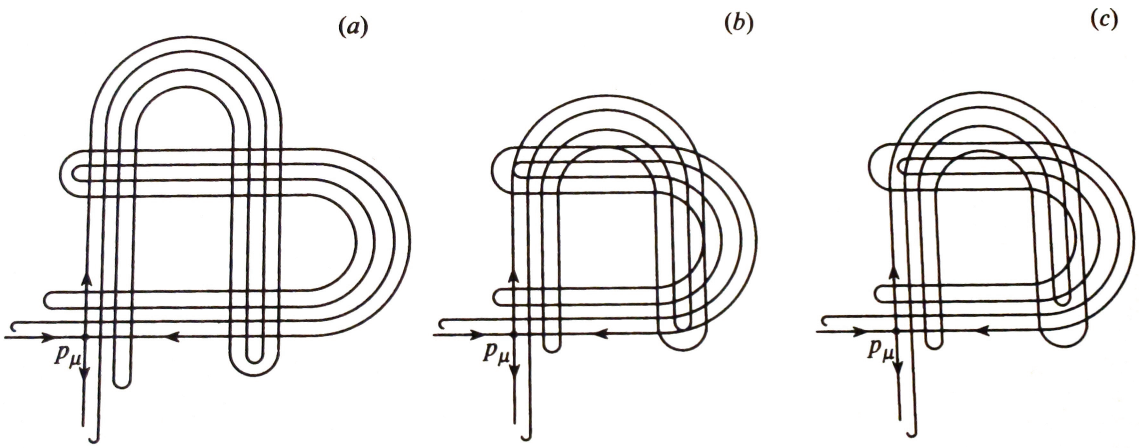

Figure 42.1: Homoclinic bifurcation ([Fig. 1.6 of Palis de Melo]

If we take a small rectangle R near p, then R has invariant foliations, and crossing points of these foliations make an invariant set that is a Cantor set. Poincare knew that transversal homoclinic points are accumulation points of other homoclinic points. Birkhoff showed that a transversal homoclinic point is an accumulation point of periodic orbits.

42.3 Cascade of homoclinic tangency

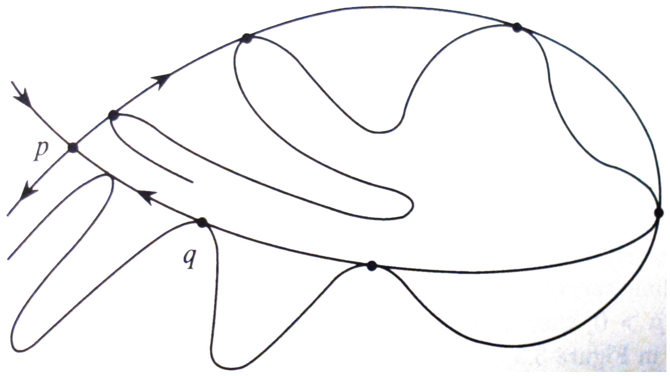

Figure 42.2: Homoclinic tangency [Palis-Takens Fig. 3.1]

Let φ_μ be a one-=parameter family of diffeomorphisms with a quadratic homoclinic tangency q at $\mu = 0$ associated to te fixed (periodic) saddle p and suppose it unfolds generically. Then, there is a sequence $\mu_n \to 0$ such that φ_{μ_n} has homoclinic tangencies $q_{\mu_n} \to q$ associated to $p_{\mu_n} \to p$.

42.4 Measure of homoclinic bifurcation set

We are interested in the measure of the set of μ in which φ_μ has a hyperbolic limit set. If the sum of the Hausdorff dimensions of W^s and W^u is less than 1, we can say this set has a relatively large Lebesgue measure.

Let B be the set of μ such that φ_μ is at a bifurcation point (homoclinic tangency). Then,[445]

$$\lim_{\mu_0 \to \mu} \frac{m(B \cap [0, \mu_0])}{\mu_0} = 0. \tag{42.1}$$

42.5 Newhouse phenomenon

Newhouse proves:
Let $\varphi \in \mathrm{Diff}^2(M)$, M a 2-manifold, be with a saddle point p whose stable and unstable manifolds have an orbit of tangency. Then,
(1) arbitrarily near φ there is an open set $U \subset \mathrm{Diff}^2(M)$ with persistent homoclinic tangencies.
(2) If moreover $|\det (d\varphi)_p| < 1$ (i.e., dissipative), then there is a residual set $R \subset U$ such that each member of which has infinitely many hyperbolic sinks.

This means that hyperbolic systems are not dense in $\mathrm{Diff}^2(M)$, M a 2-manifold.

[445] Theorem 2 of Palis-Takens p101.

However, for Diff$^i(M)$ nothing is known.

42.6 Persistent homoclinic tangency

Basic set: it is a maximal invariant set in its local) and canonical coordinate system may be taken if two points x y in it is close.
Persistent tangency: Let Λ_1 and Λ_2 are basic sets of φ. For any $\varphi \in U \subset \text{Diff}^2(M)$, for $x_1 \in \Lambda_1$ $x_2 \in \Lambda_2$ there is a tangency between $W^s(x_1)$ and $W^u(x_2)$ or C^2-close φ'.
Thickness of a Cantor set: nongap length/gzp length $+ \tau$. This is not the Hausdorff dimension.
Overlaps of K_1 and K_2. If $\tau(K_1)\tau(K_2) > 1$ then $K_1 \cap K_2 \neq \phi$ because K's are closed.

Proposition 1
Let $\varphi \in \text{Diff}^2(M)$ wutgh basic sets Λ_1 and Λ_2, both of saddle type, and let $p_i \in \Lambda_i$ be periodic points. Assume $\tau(L_1)\tau(L_2) > 1$[446] (this condition will be explained **42.9**) in and there is a orbit of tangency of $W^u(p_1)$ and $W^s(p_2)$. Then φ is in the closure of some $U \subset \text{Diff}^2(M)$, where U has persistent tangencies involving $\Lambda_1(\tilde{\varphi})$ and $\Lambda_2(\tilde{\varphi})$ of Λ_1 and Λ_2 for $\varphi \in U$.

42.7 Persistence proof

Take $\tilde{\varphi}$ near φ, two basic sets $\Lambda_1(\tilde{\varphi})$ and $\Lambda 2(\tilde{\varphi})$ and periodic points $p_1(\tilde{\varphi})$ and $p_2(\tilde{\varphi})$. They depend on $\tilde{\varphi}$ continuously.

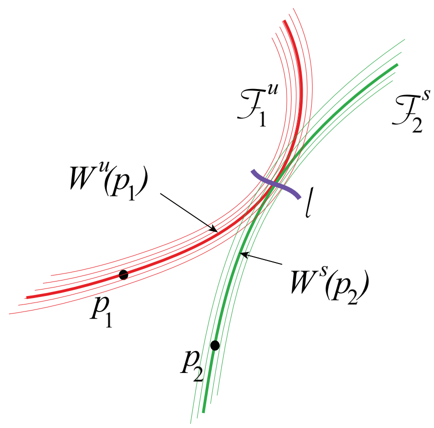

Figure 42.3:

We assume $W^u(p_1)$ and $W^s(p_2)$ are quadratically tangent (if needed, we can per-

[446] $\tau^u(\Lambda_1, p_1)\tau^s(\Lambda_2, p_2) > 1$ more preciesely in the original book.

turb the original system to enforce this).

We also have stable and unstable foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ (recall a horseshoe). All these structures depend continuously on $\tilde{\varphi}$. If we change $\varphi(\varphi)$ the tangent point changes (C^1 change) its position transversally to the foliations. We can project leaves onto this trajectory ℓ of the tangent point.'

Now, take one-sided neighborhood boxes K_1 and K_2 as in Fig. 42.4. Then, we can consider their projections along the foliation onto ℓ. Generally, the projected images are Cantor sets.

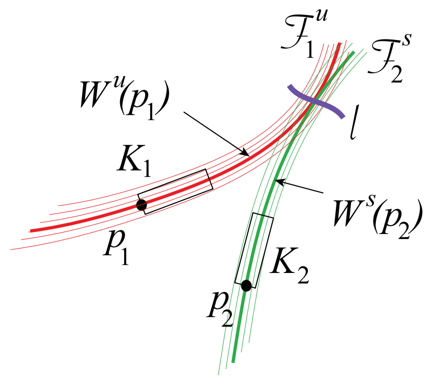

Figure 42.4:

These Cantor sets changes continuously with $\tilde{\varphi}$. We claim that under the condition "$\tau^u(\Lambda_1, p_1)\tau^s(\Lambda_2, p_2) > 1$", they persist to overlap ('really' as sets).

42.8 $\boldsymbol{\lambda}$-lemma

If ℓ is a smooth curve intersecting $W^s(p)$ transversally, then its forward image $\ell^i = \varphi^i(\ell)$ contain compact arcs $m_i \subset \ell^i$ which approaches differentiably a compact arc m in $W^u(p)$ as in Fig. 42.5.

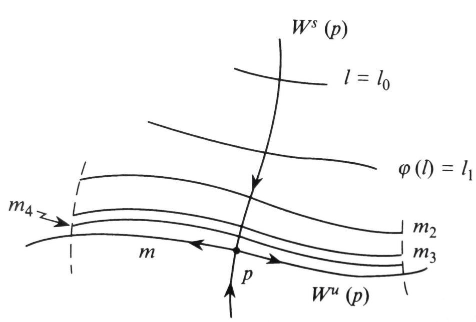

Figure 42.5: λ-lemma [Palis-Takens Fig. 1.7]

42.9 Linking Cantor sets

Suppose two Cantor sets 'generally' overlap (technical term = linked). Is there any actual overlap of points?

We define the thickness of a Cantor set K. A gap of K is a connected component of K^c. Let U be a bounded gap, and u its boundary point.

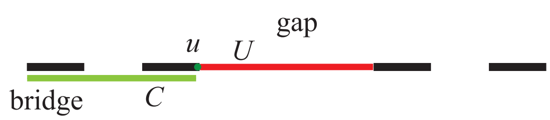

Figure 42.6: Gaps and bridges of a Cantor set

A bridge C of K at u is the maximal interval that does not contain any gap whose length is not equal or larger than U.

The thickness $\tau(K,u)$ of K at u is defined by ℓ is te length.

$$\tau(K,u) = \ell(C)/\ell(U). \tag{42.2}$$

Then, the thickness $\tau(K)$ of K is defined as

$$\tau(K) = \inf_u \tau(K,u). \tag{42.3}$$

Gap lemma:[447] If K_1 and K_2 are two Cantor sets. If

$$\tau(K_1)\tau(K_2) > 1, \tag{42.4}$$

then $K_1 \cap K_2 \neq \emptyset$, unless one of them is not engulfed in on of the other's gap.
[Explanation] (42.4) means (see Fig. 42.7)

$$\frac{\ell(C_1)}{\ell(U_1)}\frac{\ell(C_2)}{\ell(U_2)} > 1. \tag{42.5}$$

As seen in Fig. 42.7 if (42.4) holds, then we see overlaps may occur. Suppose there is certainly an overlap between some C and C' from the both Cantor sets. If there is not common point for $C \cap K$ and $C' \cap K'$, then all the points must be in U or U'. However, U and U' both shrink to zero...

[447] p63 of Palis and Takens.

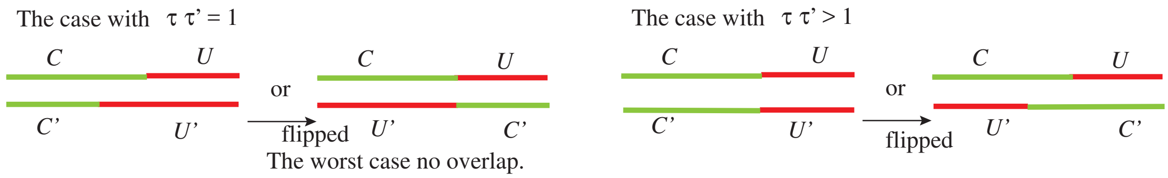

Figure 42.7: Meaning of (42.4).

42.10 Completion of 42.7

Let L_i be the projection of K_i in **42.7** onto *ell* along the foliations. These Cantor sets and the image of Ki with the dynamics (forward or backward) need not agree in general.

However, note that if the map is close to 'scaling' (i.e., the ratio of the max and min derivatives is close top unity), then the thickness is preserved. Therefore, we take K_i small enough and $\tilde{\varphi}$ is close enough to φ, this is realized.

Since L_1 and L_2 have a common set, and it consists of boundary points of Cantor sets (inevitably, since they do not have no internal point), so we have homoclinic tangency for $\tilde{\varphi}$.

42.11 Infinitely many hyperbolic sinks

Proposition 2.
Let $U \subset \text{Diff}^2(M)$ be an open set with persistent homoclinic tangencies, associated with a basic set $\Lambda(\varphi)$. Let $p(\varphi) \in \Lambda(\varphi)$ be a periodic point, say of period k and let $|\det{(d\varphi^k)_{p(\varphi)}}| < 1$. Then, there is a residual subset $R \subset U$ such that each $\varphi \in R$ has infinitely many hyperbolic periodic attractors (sinks). If $|\det{(d\varphi^k)_{p(\varphi)}}| > 1$, one gets infinitely may periodic repellers (sources).

The strategy to prove this proposition is to show that if φ has n hyperbolic sinks, then in its any neighborhood is a map with $n+1$ such sinks.

42.12 Demonstration of $n + 1$ sinks

Suppose φ has n hyperbolic sinks. They are stable against perturbations. $W^u(p)$ and $W^s(p)$ are both dense in $W^u(\Lambda)$ and $W^s(\Lambda)$ (resp.), and $W^u(\Lambda)$ and $W^s(\Lambda)$ have tangencies. Therefore, with (if needed) small perturbation we can make $W^u(p)$ and $W^s(p)$ in tangency. Let q be a point in this tangency orbit. This q is not among the already existing n periodic orbits, so we can take a neighborhood W of q that excludes n-periodic orbits.

Now, with an arbitrarily small perturbation, we can make a hyperbolic sink in W.

This can be understood from the horseshoe bifurcation **42.13**.

42.13 How does a horseshoe appear?[448]

As seen in Fig. 42.8 initially, there is no fixed point in R, but there is a horseshoe with all its hyperbolic fixed points at the 'right end.'

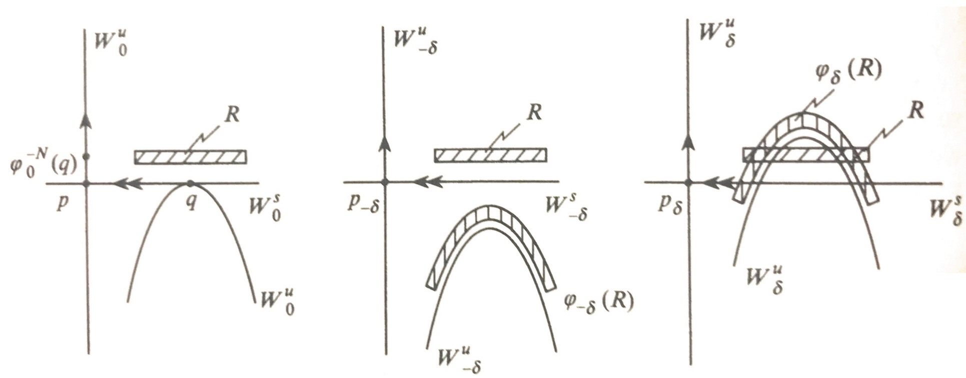

Figure 42.8: Horseshoe is closely related to the logistic map

What happens in between? It is the 'standard' story of bifurcations. We know there are three kinds of generic bifurcations in a one-parameter family of diffeomorphism; saddle-node, period doubling and Hopf. In the present context, the eigenvalues around fixed points are real, so we must consider the former two possibilities. Thus, sinks show up.

42.14 Hénon-like diffeomorphism

The Hénon map[449] $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$x_{n+1} = y_n + 1 - ax_n^2, \tag{42.6}$$

$$y_{n+1} = bx_n. \tag{42.7}$$

where $a = 1.4$ and $b = 0.3$ is the original choice of the parameters. This is written as the composition of the following three maps $T''' \circ T'' \circ T'$ (Fig. 42.9):

$$T' : \qquad x' = x, y' = y + 1 - ax^2, \tag{42.8}$$

[448] J. A. Yorke and K. T. Alligood, Cascades of period-doubling bifurcations: a prerequisite for horseshoe, Bull AMS 9 319 (1983); C. Robinson, Bifurcation to infinitely many sinks, Comm Math Phys 90 433 (1983) contain useful concrete examples.

[449] M. Hénon, A Two-dimensional Mapping with a Strange Attractor, CMP 50 69 (1976).

$$T'' : \qquad x'' = bx', y'' = y', \tag{42.9}$$
$$T''' : \qquad x''' = y'', y''' = x''. \tag{42.10}$$

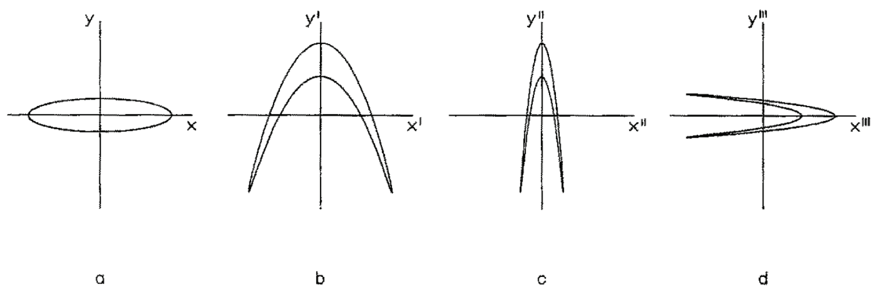

Figure 42.9: The Hénon map (d) is given by $T''' \circ T'' \circ T'$. [Fig. 1 of Hénon CMP 50 69 (1976)]

There are two fixed points:

$$x = \frac{1}{2a}\left[-1(1-b) \pm \sqrt{(1-b)^2 + 4a}\right], y = bx. \tag{42.11}$$

These points are real for $a > a_0 = (1-b)^2/4$. One is always a hyperbolic source, while the other is unstable for

$$a > a_1 = 3(1-b)^2/4. \tag{42.12}$$

Notice that not all the initial conditions give bounded orbits; they escape to infinity. The remaining set seems to be a Cantor set $\times$ smooth curves (locally). It is well-illustrated in
https://www.youtube.com/watch?v=42oeboRGqTo.

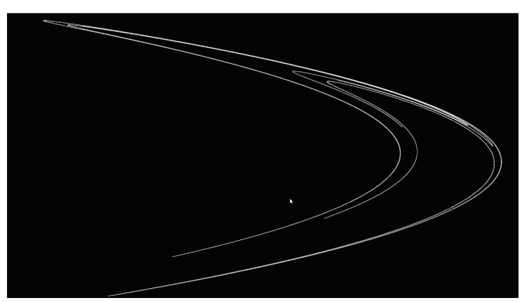

Figure 42.10: from the same YouTube.

See Michael Benedicks, Lai-Sang Young: Sina-Bowen-Ruelle measures for certain Hénon maps, Inventiones Mathematicae 112 541 (1993).